11/07/17

MATH 508

Lecture 19

Local I.F.T (inverse function thm): Suppose that f is differentiable near to a point x_0 and that f^{-1} is continuous at x_0. If $f'(x_0) \neq 0$, then there is a $\delta > 0$ s.t. f is invertible in the interval $(f(x_0)-\delta, f(x_0)+\delta)$ and the inverse is diff'l.

Pf: $f'(x_0) \neq 0$

$\Rightarrow$ f is either strictly increasing or strictly decreasing at x_0

$\Rightarrow$ $f'(x_0) \neq 0$ at x_0 in the interval $(x_0-\eta, x_0+\eta)$

Now we can apply the inverse function thm.

$y_0 = f(x_0)$, $\exists\ \delta > 0$ s.t. for $y \in (y_0-\delta, y_0+\delta)$ the ~~one~~ eqn $f(x) = y$ has a unique soln near x_0

and $f^{-1}(y)$ is a diff'l function

→ Linear approximations are good, but can we do better?

Can we find a polynomial of 2^{nd} degree such that

$$f(x) - \left(f(x_0) + f'(x_0)(x-x_0) + \alpha(x-x_0)^2 \right) = o(|x-x_0|^2)$$

This problem is solvable when f has two derivatives, i.e. f' is itself a differentiable function at x_0.

In this context, we can think of a derivative as mapping $f \to f'$, i.e. a function mapped to another function.

We say that f is twice differentiable if f' is also differentiable

$$f(x) \to \frac{df}{dx}(x)$$

$$f(x) \to D_x f(x)$$

$$f(x) \to \partial_x f(x)$$

$$f''(x) = (f'(x))'$$

$$\frac{d^2 f(x)}{dx^2} = \frac{d}{dx}\left(\frac{df}{dx}(x)\right)$$

$$\partial_x^2 f = \partial_x(\partial_x f)$$

- Higher order derivatives

$$\frac{d^n f}{dx^n}(x) = \frac{d}{dx}\left(\frac{d^{n-1}}{dx^{n-1}} f(x)\right) \longleftarrow \text{recursive defn}$$

eg: $p(x) = ax^2 + bx + c$ $\qquad p(0) = c$

$p'(x) = 2ax + b$ $\qquad p'(0) = b$

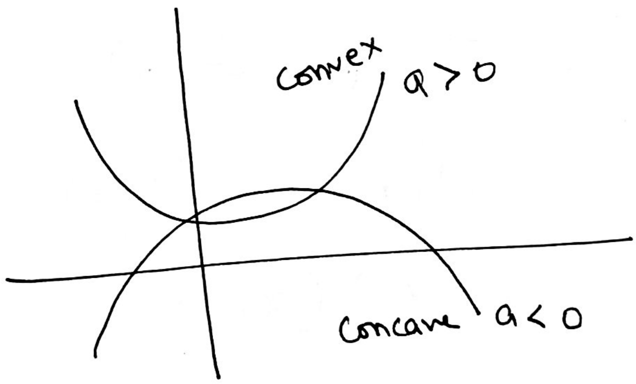
$p''(x) = 2a$ $\qquad p''(0) = 2a$

$p'''(x) = 0$

$$\boxed{p(x) = \frac{p''(0)}{2}x^2 + p'(0)x + p(0)}$$

Graphing a quadratic polynomial

only of two "types"



$\cancel{g_2(x)}$

$$\boxed{f(x) = \underbrace{\frac{f''(x)}{2}(x - x_0)^2 + f'(x_0)(x - x_0) + f(x_0)} + o(|x - x_0|^2)}$$

$g_2(x) = f(x) - o(|x - x_0|^2)$

Thm: If f is twice diff'l in an interval around x_0 then

$$F(x) = f(x) - g_2(x) = o(|x-x_0|^2)$$

Use MVT twice

$F(x_0) = 0$, $F'(x) = f'(x) - g_2'(x)$

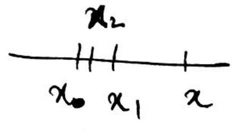
$F'(x_0) = 0$, $F''(x) = f''(x) - g_2''(x)$

$F''(x_0) = 0$

$$F(x) = F(x) - F(x_0)$$
$$= F'(x_1)(x-x_0)$$ for some x_1 that lies in b/w x and x_0

$$F'(x_1) = F'(x_1) - F'(x_0)$$
$$= F''(x_2)(x_1-x_0)$$

$$F(x) = F''(x_2)(x_1-x_0)(x-x_0)$$



$$|F(x)| \leq F''(x_2)|x-x_0|^2$$

As $x \to x_0$, $\lim_{x\to x_0} F''(x_0) = 0$

$$\Rightarrow F(x) = o(|x-x_0|^2)$$

Information in the derivatives

1^{st} derivative — slope of the tangent line

2^{nd} derivative — local convexity of the graph

Thm: Suppose that f is diff'l in a neighborhood of x_0, and $f''(x_0)$ exists

(i) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a strict local minimum for f, and if $f''(x_0) < 0$ then x_0 is a strict local max^m for f.

If $f''(x_0) > 0$, then

f' is strictly monotone increasing in a neighborhood of x_0. $\exists\, \delta > 0$

$$f'(x_0) = 0 \Rightarrow \begin{cases} f'(x) < 0 & \text{in } (x_0-\delta, x_0) \\ f'(x) > 0 & \text{in } (x_0, x+\delta) \end{cases}$$

MVT, $f(x) - f(x_0) = f'(x_1)(x - x_0)$

$f(x) - f(x_0) > 0$ if $x \in (x_0-\delta, x_0+\delta)$

$\Rightarrow x_0$ is a local strict min.

(ii) If x_0 is a local min, then $f''(x_0) \geq 0$

x_0 is a local max then $f''(x_0) \leq 0$

proof by contradiction.

Suppose x_0 is a local min and

$\underbrace{f''(x_0) < 0}$ $\Rightarrow$ x_0 is a strict local max,

but x_0 was a local min

$\times$

let $g_1(x) = f(x) + f'(x)(x - x_0)$

$F(x) = f(x) - g(x)$

If $F''(x_0) = f''(x_0) > 0$, then

by the previous result $F(x)$ has a strict local min^{m} at x_0, i.e. $f(x) > g_1(x)$ for $x \in (x_0 - \delta, x_0 + \delta)$

Thm Let f be twice differentiable on an interval (a,b) and let $a < c < d < b$, and assume that $f''(x) > 0$ in $[c,d]$

$$h(x) = f(c)\,\frac{x-d}{c-d} + f(d)\,\frac{x-c}{d-c}$$

The graph of $h(x)$ is the line from $(c, f(c))$ to $(d, f(d))$

$$f(x) < h(x) \quad \text{for } x \in (c,d)$$

Pf:

$$F(x) = f(x) - h(x)$$

$$F(c) = F(d) = 0$$

$$F''(x) = f''(x) > 0 \quad \text{in } (c,d)$$

If $\exists$ a pt $x \in [c,d]$ $F(x) \geq 0$ then F achieves a local max at ~~one~~ some point x_0 in (c,d), and then $F''(x_0) \leq 0$, a contradiction.

$$\Rightarrow F(x) < 0 \quad \Rightarrow \quad f(x) < h(x) \text{ in the interval.}$$

$\blacksquare$

Difference Quotient :

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{(x_0+h) - x_0}$$

$$f''(x_0) = \lim_{h\to 0} \frac{f'(x_0+h) - f'(x_0)}{h}$$

$$f'(x_0+h) \approx \frac{f(x_0+2h) - f(x_0+h)}{h}$$

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

Is $f''(x_0) = \lim_{h\to 0} \dfrac{\dfrac{f(x_0+2h) - f(x_0+h)}{h} - \dfrac{f(x_0+h) - f(x_0)}{h}}{h}$?

let $g(x) = f(x+h) - f(x)$

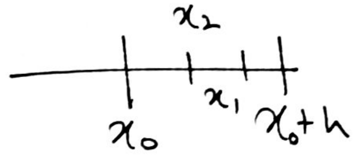
$$f''(x_0) = \lim_{h\to 0} \frac{\overbrace{f(x_0+2h) - 2f(x_0+h) + f(x_0)}^{Nr =: \Delta_h^2 f(x_0)}}{h^2}$$

$$\Delta_h^2 f(x_0) = g(x_0+h) - g(x_0)$$

Now $\left[\dfrac{\Delta_h^2 f(x_0)}{h^2} = \dfrac{g(x_0+h) - g(x_0)}{h^2}\right]$ as $h \to 0$

$$= \frac{g'(x_1)\cdot h}{h^2} \qquad x_1 \in (x_0, x_0+h)$$

$$= \frac{f'(x_1+h) - f'(x_1)}{h}$$



$$= \cancel{f''(x_0+h)} \quad f''(x_2)$$

As $h \to 0$
$x_2 \to x_1$

Thus we have, if f'' is cont. at x_0, then

$$\boxed{f''(x_0) = \lim_{h\to 0} \frac{\Delta_h^2 f(x_0)}{h^2}}$$

If f is twice diff'l $f'(x_0) = 0$, $f''(x_0) = 0$
then this gives no information of the graph ~~at~~ of f at x_0.

Defn: Inflexion point (x_0).
$f''(x) > 0$ on one side of x_0
and $f''(x) < 0$ on the other side

Obs :

If $f(x) = \sum_{j=0}^{n} a_j x^j$

$$f'(x) = \sum_{j=1}^{n} j a_j x^{j-1}$$

$$f^{(k)}(x) = \sum_{j=k}^{n} a_j j \cdot (j-1) \cdots (j-k+1) x^{j-k}$$

$$f^{(n)}(x) = n! a_n$$

$$f^{(m)}(x) = 0 \qquad m > n.$$

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}$$