11/07/17

MATH 508

Lecture 20

$\sum_{j=0}^{n} a_j x^j \leftarrow$ n^{th} degree polynomial.

A function with n derivatives is one s.t.

$\partial_x^j f(x)$ is defined for $j = 1, \ldots, n$

$$\boxed{T_n(f(x); x_0) = \sum_{j=0}^{n} \frac{f^j(x_0)}{j!}(x-x_0)}$$

— n^{th} order Taylor polynomial of f at x_0

$$f(x) - T_n(f(x); x_0) = o(|x-x_0|^n)$$

If the n^{th} derivative is continuous, then we say that f is n-times continuously diff'le

The set of such functions on $[a,b]$ is denoted

$$C^n([a,b])$$

The set is a vector

① If $f, g \in C^n([a,b])$, then so does
$\alpha f + \beta g$, $\forall \alpha, \beta \in \mathbb{R}$

② $0 \in C^n([a,b])$

③ If $f \in C^n([a,b])$, $-f \in C^n([a,b])$

<u>Thm</u> : Let $f \in C^n([a,b])$ then $\forall x, x_0 \in (a,b)$

~~then~~ $f(x) = T_n(f(x); x_0) + o(|x-x_0|^n)$

<u>Pf</u>: $F(x) = f(x) - T_n(f(x); x_0)$

$$F(x_0) = F'(x_0) = \cdots = F^n(x_0) = 0$$

$$F(x) = F(x) - F(x_0)$$

$$= F'(x_1)(x_1 - x_0) \qquad x_1 \in (x_0, x)$$

$$F'(x_1) = F'(x_1) - F'(x_0) = F''(x_2)(x_1 - x_0)$$

$$x_2 \in (x_0, x_1)$$

$\vdots$

$$F^j(x_j) = F^j(x_j) - F^j(x_0) = F^{j+1}(x_{j+1})(x_j - x_0)$$

$$x_{j+1} \in (x_0, x_j)$$

$$F^{n-1}(x_{n-1}) = F^{n}(x_n)(x_{n-1}-x_0)$$

$$F(x) = \underbrace{F'(x_1)}(x-x_0)$$

$$= F''(x_2)(x_1-x_0)(x-x_0)$$

$$\vdots$$

$$= F^{n}(x_n)(x_{n-1}-x_0)\cdots(x_1-x_0)(x-x_0)$$

$$|F(x)| = |f(x) - T_n(f(x);x_0)|$$

$$\leq |F^{n}(x_n)|\,\underbrace{|(x_{n-1}-x_0)|\cdots|(x-x_0)|}_{\text{each term } \leq |(x-x_0)|}$$

$$\Rightarrow |f(x) - T_n(f(x);x_0)| \leq |F^{(n)}(x_n)|\cdot|(x-x_0)|^{n}$$

$F^{n}(x_0) = 0$; $x_n \in (x_0, x) \Rightarrow$ As $x \to x_0$,

$x_n \longrightarrow x_0$

and $\lim_{x\to x_0} F^{n}(x_n) = 0$

$|F^{(n)}(x_n)|\cdot|(x-x_0)|^{n} = o(|x-x_0|^{n})$ — from defn.

$$f(x) - T_n(f(x); x_0) = R_n(f(x); x_0) \leftarrow n^{th} \text{ remainder term}$$
$$= o(|x - x_0|^n)$$

Lagrange's remainder formula

$$R_n(f(x); x_0) = \frac{f^{n+1}(x_1)}{(n+1)!}(x - x_0)^{n+1}$$

f has a $(n+1)^{th}$ derivative at x_0
and at $x_1 \in (x_0, x)$.

Eg

$\partial_x \sin x = \cos x$

$\partial_x \cos x = -\sin x$

$|\partial_x^n \sin x| \leq 1 \quad , \forall n$

$$|R_n(f(x); x_0)| \leq \frac{1}{(n+1)!}(x - x_0)^{n+1}$$

want 10 digits of accuracy to calculate $\sin x$

$$\frac{1}{(n+1)!} 10^{(n+1)} \leq 10^{-10} \quad , \text{ i.e. } \frac{10^{10-(n+1)}}{(n+1)!} \leq 1$$

put $n=6$ $\Rightarrow$ $\frac{1000}{5040} \leq 1$!

2^{nd} MVT :

Let f, g be cont. on $[a, b]$, diff'l on (a, b) s.t. $g(a) \neq g(b)$. Then $\exists\ \xi \in (a, b)$ s.t.

$$\boxed{f'(\xi) = g'(\xi) \frac{f(b) - f(a)}{g(b) - g(a)}}$$

$$F(x) = f(x) - \frac{g(b) - g(x)}{g(b) - g(a)} f(a) - \frac{g(x) - g(a)}{g(b) - g(a)} f(b)$$

$$F(a) = F(b) = 0$$

$\exists\ \xi \in (a, b)$ s.t.

$$F'(\xi) = 0$$

$$F'(\xi) = f'(\xi) + \frac{g'(\xi) f(a)}{g(b) - g(a)} - \frac{g'(\xi) f(b)}{g(b) - g(a)}$$

If $g'(\xi) \neq 0$, $\quad \dfrac{f'(\xi)}{g'(\xi)} = \dfrac{f(b) - f(a)}{b - a} \cdot \dfrac{b - a}{g(b) - g(a)}$

~~and~~ $$\boxed{f'(\xi) = g'(\xi) \frac{f(b) - f(a)}{g(b) - g(a)}}$$

Claim

$$f(x) - T_n(f(x); x_0) = \frac{f^{n+1}(x_1)(x-x_0)^{n+1}}{(n+1)!}$$

for some $x_1 \in (x_0, x)$

Pf: Induction on n

$n = 0$

$$f(x) = f(x_0) + f'(x_1)(x-x_0) \qquad x_1 \in (x_0, x)$$

$$\boxed{\partial_x T_n(f(x); x_0) = T_{n-1}(f'(x); x_0)}$$

Assume the result for n

Consider, $$\frac{f(x) - T_{n+1}(f(x); x_0)}{(x-x_0)^{n+2}}$$

$$(2^{nd}\ \text{MVT}) = \frac{f'(\xi) - \partial_x T_{n+1}(f(\xi), x_0)}{(n+2)(\xi - x_0)^{n+1}} \qquad \xi \in (x_0, x)$$

$$= \frac{f'(\xi) - T_n(f'(\xi); x_0)}{(n+2)(\xi - x_0)^{n+1}}$$

$$= \frac{f^{n+2}(\eta)(\xi - x_0)^{n+1}}{(n+2)(n+1)!\,(\xi - x_0)^{n+1}} = \frac{f^{n+2}(\eta)}{(n+2)!}$$

$$\Rightarrow f(x) - T_{n+1}(f(x), x_0) = \frac{f^{n+2}(\eta)(x - x_0)^{n+2}}{(n+2)!}$$

Thm : Suppose that f and g are diff'l, defined on a neighbourhood of x_0 s.t.

(i) $f(x_0) = g(x_0) = 0$

(ii) $g'(x_0) \neq 0$ $\longrightarrow$ $g(x) = g'(x_0)(x - x_0) + o(|x - x_0|)$

and $o(|x-x_0|) \leq \frac{g'(x_0)}{2}|x-x_0|$

$\exists\, \delta > 0$, s.t. $x \in (x_0 - \delta, x_0 + \delta)$ and $x \neq x_0$, $g(x) \neq 0$

$\Rightarrow \frac{f(x)}{g(x)}$ is defined on $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$

and $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$

Pf :

$$\frac{f(x)}{g(x)} = \frac{f'(x_0)(x-x_0) + o(|x-x_0|)}{g'(x_0)(x-x_0) + o(|x-x_0|)}$$

$$= \frac{f'(x_0) + \underbrace{\frac{o(|x-x_0|)}{x-x_0}}_{o(1)}}{g'(x_0) + \underbrace{\frac{o(|x-x_0|)}{x-x_0}}_{o(1)}}$$

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x_0) + o(1)}{g'(x_0) + o(1)} \qquad \left[\lim_{x\to x_0} o(1) = 0\right]$$

$$= \frac{f'(x_0)}{g'(x_0)}$$

Q. Does $\frac{f(x)}{g(x)}$ have a derivative at x_0 ?

If f and g are twice differentiable at x_0, then $\frac{f}{g}$ has a derivative at x_0 given by

$$\frac{f'(x_0)g''(x_0) - f''(x_0)g'(x_0)}{2g'(x_0)^2}$$

$$\lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f'(x_0)}{g'(x_0)}}{x - x_0} = \frac{f(x) \cdot g'(x_0) - f'(x_0) g(x)}{g(x) \cdot g'(x_0)(x - x_0)}$$

$$Nr. = \left(f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + o(|x-x_0|)\right) g'(x_0)$$

$$- f'(x_0)\left(g'(x_0)(x-x_0) + \frac{g''(x_0)}{2}(x-x_0)^2 + o(|x-x_0|)\right)$$

$$\frac{Nr}{Dr} = \frac{\left(\frac{f''(x_0) g'(x_0) - g''(x_0) \cdot f'(x_0)}{2}\right)(x-x_0)^2 + o(|x-x_0|^2)}{g'(x_0) \cdot (g'(x_0)(x-x_0) + o|x-x_0|)(x-x_0)}$$

[$\div$ by $(x-x_0)^2$]

$$= \frac{f''(x_0) \cdot g'(x_0) - g''(x_0) f'(x_0) + o(1)}{2\, g'(x_0)^2 + o(1)}$$

Taking limit, as $x \to x_0$ gives desired answer.

f, g s.t. $f(x_0) = f'(x_0) = \dots = f^{(n-1)}(x_0) = 0$
$g(x_0) = g'(x_0) = \dots = g^{n-1}(x_0) = 0$

then and $g^n(x_0) \neq 0$

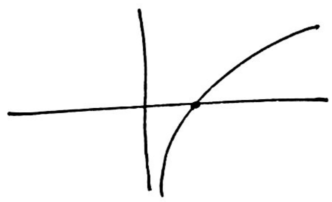
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f^n(x_0)}{g^n(x_0)}$$

Pf:

Idea: use Taylor polynomials, and follow the previous proof

Order of Vanishing

$f(x_0) = 0$, $f'(x_0) \neq 0$



$\leftarrow$ order 1 zero

Generalization: $f(x_0) = \dots = f^{n-1}(x_0) = 0$, $f^n(x_0) \neq 0$

$$f(x) = \frac{f^{(n+1)}(x_1)(x - x_0)^{n+1}}{(n+1)!}$$

This is a zero of order $(n+1)$.