11/16/17

MATH 508

Lecture 22

If f is n-times diff'l, is it true that

$$f(x) - T_n(f, x_0, x) = o(|x - x_0|^n)$$

$$f(x) = T_{n-2}(f, x_0, x) + \int_{x_0}^{x} \frac{(x-t)^{n-2}}{(n-2)!} f^{[n-1]}(t)\,dt$$

$f^{[n-1]}$ is diff'l at x_0 if

$$f^{[n-1]}(t) - f^{[n-1]}(x_0) = f^{[n]}(x_0)(t - x_0) + e(t)$$

where $|e(t)| = \cancel{o(|t-x_0|)}$ $o(|x_0 - t|)$

Now,

$$f(x) = T_{n-2}(f, x_0, x) + \int_{x_0}^{x} \frac{(x-t)^{n-2}}{(n-2)!}\left[f^{[n-1]}(x_0) + f^{[n]}(x_0)(t - x_0) + e(t)\right] dt$$

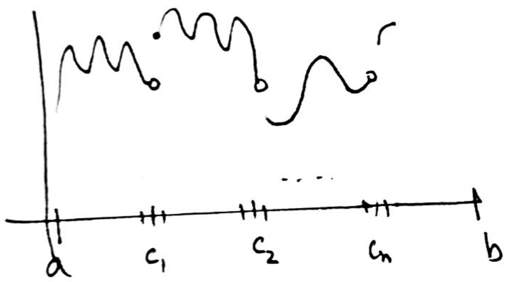
$$= T_{n-1}(f, x_0, x) + \int_{x_0}^{x} \frac{(x-t)^{n-2}(t - x_0)}{(n-2)!} f^{[n]}(x_0)\,dt + \int_{x_0}^{x} \frac{(x-t)^{n-2}}{(n-2)!} o(|x - x_0|)\,dt$$

$$= T_n(f, x_0, x) + \underbrace{\int \frac{(x-t)^{n-2}}{(n-2)!}\, o(|x-x_0|)\, dt}_{\leq\, o(|x-x_0|^n)}$$

which completes the proof.

<u>Note</u> : $O(|x-x_0|^c) \cdot o(|x-x_0|^k) = o(|x-x_0|^{c+k})$

Integrating a function with discontinuities.



$|f| \leq M$

$(c_1-\delta, c_1+\delta) \ldots (c_n-\delta, c_n+\delta)$

P is portion of $[a,b] \setminus \bigcup (c_j-\delta, c_j+\delta)$

Given $\varepsilon > 0$, $S^+(f,P) - S^-(f,P) \leq \varepsilon$

$P' = P \cup [c_j-\delta, c_j+\delta)$

$S^+(f,P') - S^-(f,P') \leq S^+(f,P) + S^-(f,P) + 2M2n\delta$

$\forall \varepsilon > 0, \exists$ partitions $\tilde{P}$ of $[a,b]$

s.t. $S^+(f,\tilde{P}) - S^-(f,\tilde{P}) < \varepsilon$

- Dirichlet's Function

$$D(x) = \begin{cases} 1 & x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & x \in \mathbb{Q} \end{cases}$$

— D is not integrable

If f is a bounded function on $[a,b]$ and P is a partition then $S^{\pm}(f,P)$ are well defined.

$$P = \{ x_0 < x_1 \; \cdots \; < x_N \}$$

$$M_j = \sup\{ f(x) : x \in [x_{j-1}, x_j] \}$$

$$m_j = \inf\{ f(x) : x \in [x_{j-1}, x_j] \}$$

$$S^+(f,P) = \sum_1 M_j (x_j - x_{j-1})$$

$$S^-(f,P) = \sum_1 m_j (x_j - x_{j-1})$$

If $P' \supseteq P$, then $S^+(f,P) \geq S^+(f,P')$
$S^-(f,P) \leq S^-(f,P')$

$\Rightarrow \forall P_1, P_2 \quad S^-(f,P_1) \leq S^+(f,P_2)$

if Y is a selection of points from P

$$S^-(f,P) \leq S(f,Y,P) \leq S^+(f,P)$$

$Y = \{y_1, \ldots, y_N\}, \quad y_j \in [x_{j-1}, x_j]$

$$\mathrm{Osc}(f, P) = S^{+}(f, P) - S^{-}(f, P)$$

We say that f is integrable if

$$\mathrm{Osc}(f, P) \longrightarrow 0$$
$$\text{as } |P| \longrightarrow 0$$

Lemma : If $|P'|$ is ~~a~~ less than the min. ~~interval~~ length of an interval in P, then

$$\mathrm{Osc}(f, P') \leq 3 \cdot \mathrm{Osc}(f, P)$$

Pf :

Let $[x'_{j-1}, x'_j] \in P'$

So $[x'_{j-1}, x'_j]$ intersects at most 2 intervals in P.

$$[x'_{j-1}, x'_j] \subseteq [x_{\ell-1}, x_\ell] \cup [x_\ell, x_{\ell+1}]$$

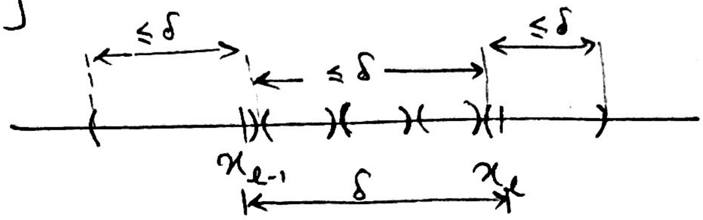
[not the same as Osc] $\leftarrow$ $$\mathrm{osc}(f, [x'_{j-1}, x'_j]) = \sup\{f(x) : x \in [x'_{j-1}, x'_j]\} - \inf\{f(x) : x \in [x'_{j-1}, x'_j]\}$$

$$M'_j - m'_j \leq (M_\ell - M_\ell) + (M_{\ell+1} - m_{\ell+1})$$

$$\sum_{j=1}^{N'} (M'_j - m'_j)(x'_j - x'_{j-1}) \leq \sum_{j=1}^{N'} \left((M_\ell - m_\ell) + (M_{\ell+1} - m_{\ell+1})\right) \cdot (x_j^{*} - x'_{j-1})$$

$$\mathrm{Osc}(f, P') \leq \sum_{\ell=1}^{N} (M_\ell - m_\ell) \sum_{\{j : [x'_{j-1}, x'_j] \cap [x_{\ell-1}, x_\ell] \neq \phi\}} (x'_j - x'_{j-1})$$

$$\sum_{\{j : [x_j', x_{j-1}'] \cap [x_{\ell-1}, x_\ell] \neq \phi\}} (x_j' - x_{j-1}') \leq 3(x_{\ell-1} - x_\ell)$$



$$\Rightarrow \text{Osc}(f, P') \leq \sum_1 (M_\ell - m_\ell) \cdot 3(x_{\ell-1} - x_\ell)$$

$$= 3 \cdot \text{Osc}(f, P)$$

■

<u>Thm</u>: Let f be a bounded function on $[a, b]$.

The following are equivalent:

(i) $\exists$ a seq. of partitions P_j such that the oscillation of f on $P_j \to 0$.

(ii) Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. if

$$|P| < \delta \text{ then } \text{Osc}(f, P) < \varepsilon$$

(iii) ~~Thee~~ $\cancel{\inf (S^+, P) = \sup}$

$$\inf_P S^+(f, P) = \sup_P S^-(f, P) = I$$

(iv) $\exists$ a seq. of partitions $\langle P_j \rangle$ and a real # I s.t. for any $\underbrace{\langle Y_j \rangle}_{\text{of pts.}} \subseteq P_j$, $S(f, Y_j, P_j) \to I$

(v) $S(f, Y, P) \to I$ as $|P| \to 0$

$\forall y$ in P, given $\varepsilon > 0$, $\exists \delta > 0$

s.t. $|S(f, Y, P) - I| < \varepsilon$

if $|P| < \delta$ for any Y in P

$$\left[S(f, Y, P) = \sum_{j=1}^{N} f(y_j) \cdot (x_j - x_{j-1}) \right]$$

<u>Pf</u>:

(ii) $\to$ (i) [Obvious]

(i) $\to$ (ii)

Choose $\langle P_i \rangle$ s.t. $Osc(f, P_i') \to 0$

can choose $|P_i| \to 0$

if P_i' is a refinement of P_i

then $Osc(f, P_i') \leq Osc(f, P_i)$

Given $\varepsilon > 0$, $\exists$ i s.t.

$Osc(f, P_i') < \varepsilon$

Now if P' is any partition s.t. $|P|$ is smaller than the smallest interval in P_i

$Osc(f, P) \leq 3\, Osc(f, P_i)$

$< 3\varepsilon$ ✓

(ii) $\Longleftrightarrow$ (iii) ✓

(iv) $\longrightarrow$ (i)

For each i choose Y_i^+, ~~s.t.~~ Y_i^- s.t.

$$S^+(f, P_i) - S(f, Y_i^+, P_i) \leq \frac{1}{2^i}$$

$$S(f, Y_i^-, P_i) - S^-(f, P_i) \leq \frac{1}{2^i}$$

$$\Rightarrow S^+(f, P_i) - S^-(f, P_i) \leq \frac{1}{2^{i-1}} + S(f, Y_i^+, P_i) - S(f, Y_i^-, P_i)$$

(i) $\longrightarrow$ (iv)

$$S^-(f, P_i) \leq S(f, Y_i, P_i) \leq S^+(f, P_i) \text{ — ✓}$$

Defn

A function ~~which~~ that satisfies these conditions is said to be RIEMANN INTEGRABLE.

Note: Not sufficient to assume $\exists \langle P_i \rangle$ with $|P_j| \to 0$ and pts. $\not\exists$ Y_i in P_i s.t. $S(f, Y_i, P_i) \longrightarrow I$.

~~Giv~~

Given f, g R.I. functions:

(1) $f+g$ is R.I.

(2) $f \cdot g$ is R.I.

(3) $c \cdot f$ is R.I. (c is a real number)

(4) $a < b < c$

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

(5) If $m \leq f \leq M$ then,

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

(6) $$\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

Need to show: $|f(x)|$ is integrable.
$\forall$ partitions P, $Osc(|f|, P) \leq Osc(f, P)$, and applying the triangle inequality $\Rightarrow$ (6).