11/30/17

MATH 508

Lecture 25

Convergence of sequences of functions

$f_n(x)$ are functions defined on $[a, b]$.

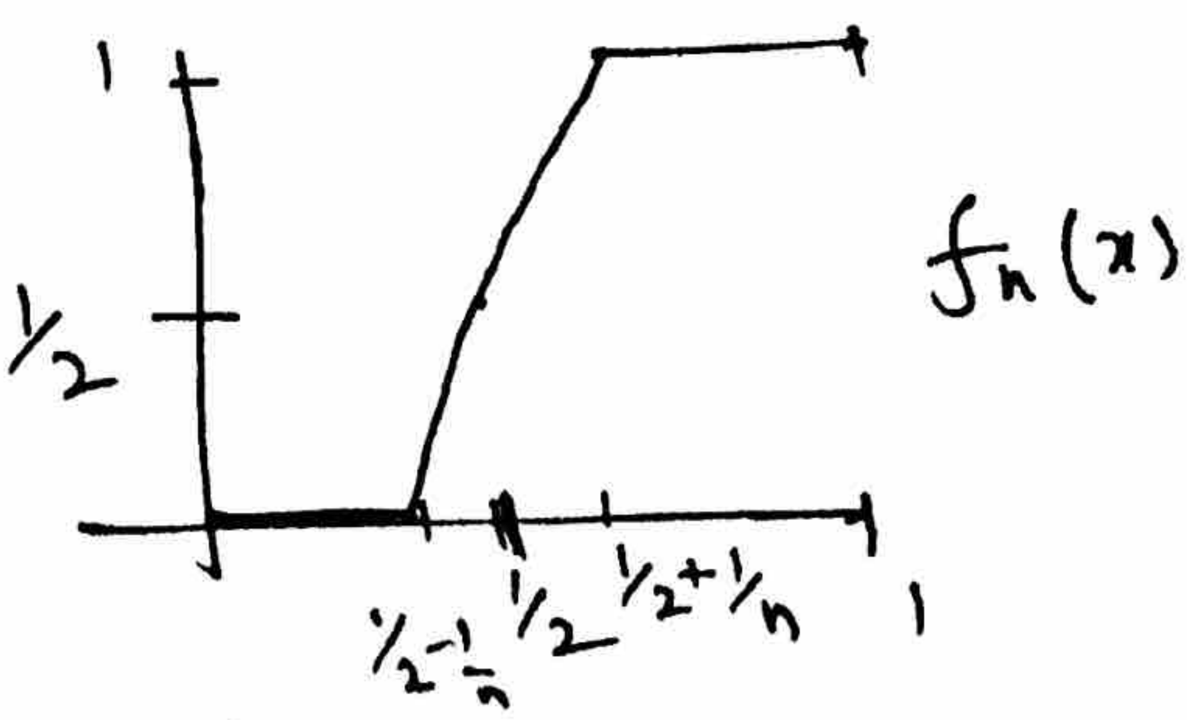
$\langle f_n(x) \rangle$

If $\forall\, x \in [a, b]$ $\lim_{n\to\infty} f_n(x) = f^*(x)$ exists

then we say $\langle f_n \rangle$ converges pointwise to f^*

Q. Suppose $f_n \in C^0([a, b])$.

Is $f^* \in C^0$?



Counter-example 1

$$\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \frac{1}{2} < x \le 1 \\ \frac{1}{2} & x = \frac{1}{2} \\ 0 & 0 \le x < \frac{1}{2} \end{cases} \longrightarrow \text{The limit is discontinuous}$$

$\langle f_n \rangle$ does not converge uniformly to f^*.

Defn: We say that f_n converges uniformly to f^* if $\forall\, \varepsilon > 0$, $\exists\, N$ s.t. $\forall\, x \in [a, b]$

$$|f^*(x) - f_n(x)| < \varepsilon \quad \text{if } n \ge N$$

Recall : $\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$

The following ~~next~~ two statements are equivalent:

(1) f_n converges uniformly to f^*

(2) $\lim_{n\to\infty} \|f_n - f^*\|_\infty = 0$

THEOREM

If $\langle f_n \rangle \subseteq C^0([a,b])$ which converges uniformly to f^*, then $f^* \in C^0([a,b])$

Pf: Given $\varepsilon > 0$, we need to show that there exists a $\delta > 0$ s.t. [use ~~uniform~~ ~~tan~~ ~~abs~~ uniform continuity as $[a,b]$ is compact]

$$|f^*(x) - f^*(y)| < \varepsilon \text{ if } |x-y| < \delta$$

Step 1: $\exists N$ s.t. if $n \geq N$ then $|f_n(x) - f^*(x)| < \frac{\varepsilon}{3}$ $\forall x \in [a,b]$

$$|f^*(x) - f^*(y)| \leq \underbrace{|f^*(x) - f_n(x)|}_{<\varepsilon/3} + |f_n(x) - f_n(y)| + \underbrace{|f^*(y) - f_n(y)|}_{<\varepsilon/3}$$

$$< \frac{2\varepsilon}{3} + |f_n(x) - f_n(y)| \quad \forall n \geq N$$

Fix $n_0 > N$

$\exists\, \delta > 0$ s.t. if $|x-y| < \delta$

then $|f_{n_0}(x) - f_{n_0}(y)| < \frac{\varepsilon}{3}$

$\Rightarrow$ $|f^*(x) - f^*(y)| < \frac{2\varepsilon}{3} + |f_{n_0}(x) - f_{n_0}(y)|$

$< \varepsilon$ if $|x-y| < \delta$

$\blacksquare$

$C^0([a,b])$ is a vector space. With $\|\cdot\|_\infty$ norm, this is a normed vector space

Q. Is $\{C^0([a,b]), \|\cdot\|_\infty\}$ a complete vector space?

Recall: A normed vector space is complete if the limit of a Cauchy sequence ~~has~~ (of terms from the vector space) has a limit in the vector space

Ans: Yes. A seq $\langle f_n \rangle \subseteq C^0([a,b])$ is a Cauchy seq. if given $\varepsilon > 0$ $\exists N$ s.t. $\|f_n - f_m\|_\infty < \varepsilon$ if $n, m \geq N$.

This assumption $\Rightarrow \forall x \in [a,b]$, the seq. of numbers $\langle f_n(x) \rangle$ is a Cauchy sequence

def of $\|\cdot\|_\infty$ $\longrightarrow$ since: $\sup_{x \in [a,b]} \{|f_n(x) - f_m(x)|\} < \varepsilon$ if $n, m \geq N$

know that: $\forall x \in [a,b]$

$$f^*(x) = \lim_{n\to\infty} f_n(x) \quad \text{exists}$$

Need to show: $\| f_n - f^* \|_\infty \longrightarrow 0$ as $n \to \infty$

I know that $\forall \varepsilon > 0 \ \exists N$ s.t. $\forall x \in [a,b]$:

$$| f_n(x) - f_m(x) | < \varepsilon \quad \text{if } n, m \geq N$$

$$\forall x \in [a,b] : | f_n(x) - f^*(x) | = \lim_{m\to\infty} | f_n(x) - f_m(x) | < \varepsilon$$

if $n \geq N$

$\Updownarrow$ $\quad \| f_n - f^* \| \leq \varepsilon$ if $n \geq N$

By the previous thm, $f^* \in C^0$.

Thus, $\{ C^0([a,b]), \|\cdot\|_\infty \}$ is a complete vector space

A diff. way to define a norm on $C^0([a,b])$ is to use

$$\| f \|_1 = \int_a^b | f(x) | \, dx$$

$$\int_a^b | f(x) + g(x) | \, dx \leq \int_a^b | f(x) | \, dx + \int_a^b | g(x) | \, dx$$

$$\| f + g \|_1 \leq \| f \|_1 + \| g \|_1$$

If $\| f \|_1 = 0$, does this $\Rightarrow f = 0$?, $f \in C^0([a,b])$

the answer is yes. If $f \neq 0$, then $\exists$ an $x_0 \in [a,b]$ s.t. $| f(x_0) | > \delta > 0$. Because f is cont. $\exists\, \eta > 0$ s.t.

$$| f(x) | > \delta/2 \text{ if } x \in \quad | [x_0 - \eta, x_0 + \eta] \cap [a,b] | \geq \eta$$

$$\int_a^b | f(x) | \, dx \geq \int_{[x_0-\eta, x_0+\eta] \cap [a,b]} \frac{\delta}{2} \, dx = \frac{\eta \delta}{2} \Rightarrow \|\cdot\|_1 \text{ defines a norm on } C^0([a,b]).$$

Q. Is $C^0([a,b])$ complete w.r.t. $\|\cdot\|_1$?

A. No. Obvious from counter-example 1 [start of this lecture]

THEOREM

Let $\langle f_n \rangle$ be R.I. functions on $[a,b]$. Suppose that $\lim_{n\to\infty} \| f_n - f^* \|_\infty = 0$, then f^* is R.I.

and $\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f^*(x)\,dx$

(Interchange of limit operations)

Pf:

Fix $\varepsilon > 0$

P is a partition of $[a,b]$, Y is a choice of points in P

$$S(f^*, Y, P) = \sum_{j=1}^{N} f(y_j)(x_j - x_{j-1})$$

$$\sup_Y S(f^*,Y,P) - \inf_Y S(f^*,Y,P) < \delta \quad \text{if } |P| < \delta$$

want to show.

Observe: $S(af + bg, Y, P) = a\,S(f,Y,P) + b\,S(g,Y,P)$

$\exists N$ s.t. if $n \geq N$ then $|f_n(x) - f^*(x)| < \varepsilon \quad \forall x \in [a,b]$

$$S(f^*,Y,P) = S(f^* - f_n, Y, P) + S(f_n, Y, P)$$

$$|S(f^* - f_n, Y, P)| \leq S(|f^* - f_n|, Y, P)$$
$$\leq \varepsilon(b-a) \quad \text{if } n \geq N$$

For an $n_0 \geq N$.

$$\sup_Y S(f^*, Y, P) = \sup_Y [S(f^* - f_{n_0}, Y, P) + S(f_{n_0}, Y, P)]$$
$$\leq \sup_Y S(|f^* - f_{n_0}|, Y, P) + \sup_Y S(f_{n_0}, Y, P)$$
$$\leq \varepsilon(b-a) + \sup_Y S(f_{n_0}, Y, P)$$

$|||^{ly}$, $\inf_Y S(f^*, Y, P) \geq -\varepsilon(b-a) + \inf_Y S(f_{n_0}, Y, P)$

$$Osc(f^*, P) \leq Osc(f_{n_0}, P) + 2\varepsilon(b-a)$$

Because f_{n_0} is R.I., $\exists$ a δ s.t. if $|P| < \delta$, then $Osc(f_{n_0}, P) < \varepsilon$

If $|P| < \delta$, then $Osc(f^*, P) < 2(b-a)\varepsilon + \varepsilon$
$$= (2(b-a)+1)\varepsilon$$
$$\Rightarrow f^* \text{ is R.I.}$$

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f^*(x)\,dx \right|$$

$$= \left| \int_a^b (f_n(x) - f^*(x))\,dx \right|$$

$$\leq \int_a^b |f_n(x) - f^*(x)|\,dx$$

$$(\text{if } n \geq N) \quad \leq \int_a^b \varepsilon\,dx = (b-a)\,\varepsilon$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f^*(x)\,dx}$$

■

Let $\{r_n\}$ be an enumeration of the points in $\mathbb{Q} \cap [0,1]$

Define:

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \{r_1, r_2, \ldots r_n\} \\ 0 & \text{ow} \end{cases}$$

f_n is R.I. for each n

$$\int_0^1 f_n(x)\,dx = 0, \quad \lim_{n\to\infty} f(x) = \underbrace{D(x)} \quad \forall x \in [0,1]$$

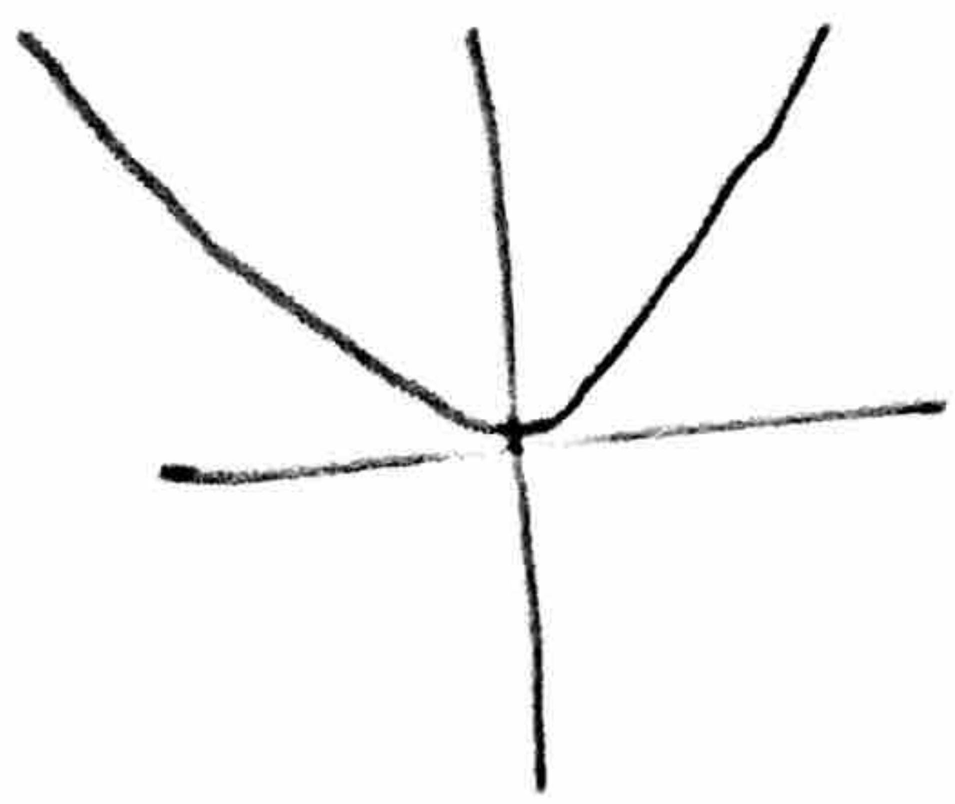
$$= \begin{matrix} 1 & x \in \mathbb{Q} \cap [0,1] \\ 0 & \text{ow} \end{matrix}$$

D is not R.I.

$\uparrow$

(Dirichlet's function)

Eg: $f_n(x) = \left(x^2 + \frac{1}{n^2}\right)^{1/2}$ ⟵ infinite diff'l



$\lim_{n\to\infty} f_n(x) = |x|$ ⟵ not diff'l at $x = 0$.

⟶ does not preserve differentiability