12/05/17

MATH 508

Lecture 26

Final Exam → Dec 18 12:00 - 2:00pm
DRL A8

Review Session → Thursday 12/~~07~~/17

Thm:
If $\langle f_n \rangle$ are R-I. on $[a,b]$ and $f_n \to f^*$ on $[a,b]$, then $\lim\limits_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f^*(x)\,dx$

Note: A uniform limit of diff'l functions is not diff'l

$$f_n(x) = \left(x^2 + \frac{1}{n^2}\right)^{1/2} \longleftarrow \text{diff'l functions}$$

$$\lim_{n\to\infty} f_n(x) = |x| \longleftarrow \text{not diff'l at } x = 0.$$

Weierstrass Appx Thm

If $f \in C^0([a,b])$, then there is a seq. of polynomials $\langle P_n \rangle$ s.t. $P_n \to f$ uniformly on $[a,b]$

(will be proved later)

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)\,dt$$

Let $\langle f_n \rangle \subseteq C^1([a,b])$

$x_0 \in [a,b]$

If $f_n(x_0) \longrightarrow \alpha$ and $f_n' \longrightarrow g$ and then we know that

$$\lim_{n\to\infty} f_n'(x) = \alpha + \int_{x_0}^{x} g(t)\,dt$$

gives conditions for limit being diff'l

THEOREM

If $\langle f_n \rangle \subseteq C'([a,b])$ has properties that

1. $f_n(x_0) \longrightarrow \alpha$ for some $x_0 \in [a,b]$
2. $f_n' \rightarrow g$ uniformly on $[a,b]$, then

$f^*(x) = \alpha + \int_{x_0}^{x} g(t)\,dt$ is the uniform limit of $\langle f_n \rangle$. f^* is C' and $\frac{df^*}{dx} = g = \lim_{n\to\infty} f_n'$

Pf:

By the Fundamental Th of Calculus,

$$f^* \in C^1([a,b]) \quad \text{with} \quad \frac{df^*}{dx} = g$$

$$(f_n - f^*)(x) = f_n(x_0) - \alpha + \int_{x_0}^{x} (f_n'(t) - g(t))\,dt$$

$$|f_n(x) - f^*(x)| \leq |f_n(x_0) - \alpha| + \int_{x_0}^{x} |f_n'(t) - g(t)|\,dt$$

Given $\varepsilon > 0$, $\exists N$ st

$$|f_n(x_0) - \alpha| < \varepsilon \quad \forall n \geq N$$

$f_n' \to g$ so $\exists N'$ s.t. if $n \geq N'$, then

$$|f_n'(t) - g(t)| < \varepsilon \quad \forall t \in [a,b]$$

Let $n \geq \max(N, N')$

$$\Rightarrow |f_n(x) - f^*(x)| \leq \varepsilon + \int_{x_0}^{x} \varepsilon\,dt$$

$$\leq (1 + (b-a))\varepsilon$$

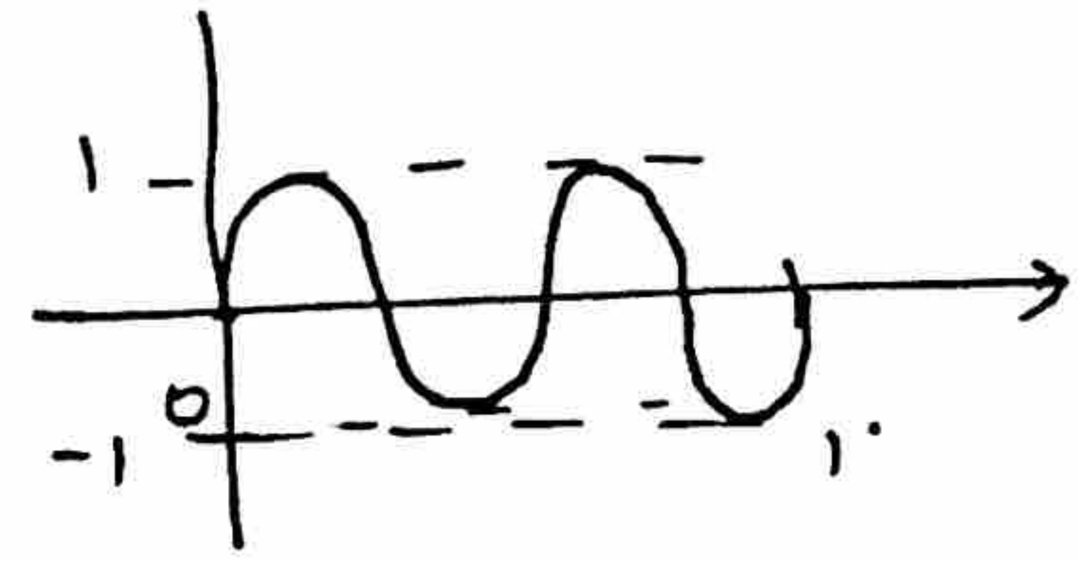
$$\Rightarrow f_n(x) \to f^*(x)$$

∎

THEOREM

Let $f \in C^0([a,b])$

$$\lim_{n\to\infty} \int_0^1 f(x) \sin n\pi x \, dx = 0$$



Pf:

Consider $\left| \int (f(x) - g(x)) \sin n\pi x \, dx \right| \leq \int_0^1 |f(x) - g(x)| \sin n\pi x \, dx$

$$\leq \int_0^1 |f(x) - g(x)| \, dx \leq \|f - g\|_\infty$$

Suppose we want to show that

$$\lim_{n\to\infty} \int_0^1 f(x) \sin n\pi x \, dx = 0$$

It's enough to show that

$$\forall \varepsilon > 0$$

$$\limsup_{n\to\infty} \left| \int_0^1 f(x) \sin n\pi x \, dx \right| < \varepsilon$$

and say we can find a function $g(x)$ s.t.

$\|f - g\|_\infty \leq t$ and

$$\lim_{n\to\infty} \int_0^1 g(x) \sin n\pi x \, dx = 0$$

$$\int_0^1 f(x)\sin n\pi x\,dx = \int_0^1 (f(x)-g(x))\sin n\pi x\,dx + \int_0^1 g(x)\sin n\pi x\,dx$$

$$\left|\int_0^1 f(x)\sin n\pi x\,dx\right| \leq \int_0^1 |f(x)-g(x)| \sin n\pi x\,dx + \left|\int_0^1 g(x)\sin n\pi x\,dx\right|$$

$$\leq \varepsilon + \underbrace{\left|\int_0^1 g(x)\sin n\pi x\,dx\right|}_{\longrightarrow 0 \text{ as } n\to\infty}$$

$$\Rightarrow \limsup_{n\to\infty} \left|\int_0^1 f(x)\sin n\pi x\,dx\right| \leq \varepsilon$$

Consider: $\underbrace{\int_0^1 g(x)\sin n\pi x}_{\text{(Int. by parts)}}$, $g \in C^1([0,1])$

$$-\frac{g(x)\cos n\pi x}{n\pi}\Big|_0^1 + \int_0^1 \frac{g'(x)\cos n\pi x}{n\pi}$$

$$\left|\int_0^1 \frac{g'(x)\cos n\pi x}{n\pi}\right| \leq \frac{\|g'\|_\infty}{n\pi} \Rightarrow \boxed{\left|\int_0^1 \underline{g'(x)\,\cancel{\sin}\, n\pi x}\,dx\right| \leq c/n}$$

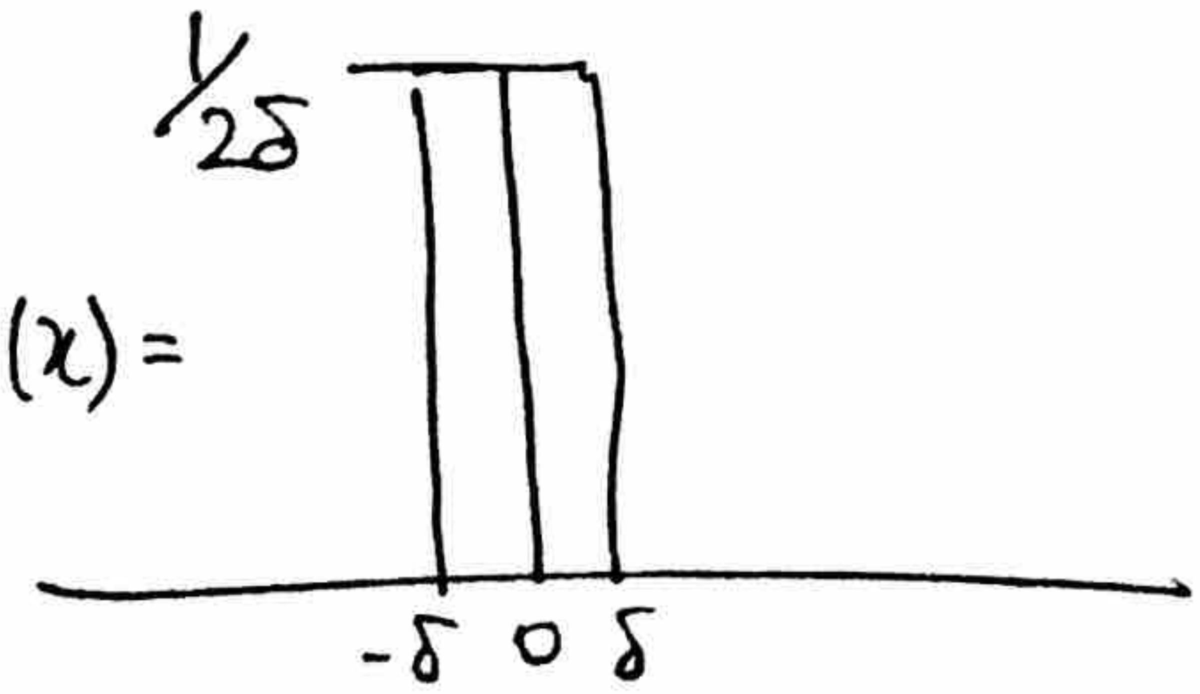
class of func. for approximating →

Convolution is an operation that takes a pair of of functions to a function.

Let f, g be R-I., and suppose that one vanishes outside of a bounded interval $[a, b]$.

We define

$$f * g(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy$$

$g(x) =$ 

$$\int_0^1 g(x)\, dx = 1$$

$$f * g(x) = \int f(y)\, g(x-y)\, dy$$

$\cancel{-\delta \le x-y \le \delta}$ $\quad -\delta \le x-y \le \delta$

$$x-\delta \le y \le x+\delta$$

$$= \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(y)\, dy$$

$g_\delta := g$

$$\frac{d}{dx}\left(f * g_\delta(x)\right) = \frac{f(x+\delta) - f(x-\delta)}{2\delta}$$

If f is not diff'l at x then

$$\lim_{\delta \to 0} \frac{d}{dx}\left(f * g_\delta(x)\right) \text{ does not exist}$$

$$\left(f * g_\delta(x)\right) - f(x) = \int \left(f(y) - f(x)\right) g_\delta(x-y)\, dy$$

$$= \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} \left(f(y) - f(x)\right) dx$$

Fix $[a,b]$ & an $\varepsilon > 0$

since f is continuous on $[a,b]$. This means that there is a $\delta_0 > 0$ s.t. if $|x-y| < \delta_0$ then

$|f(x) - f(y)| < \varepsilon$

Choose $\delta < \delta_0$ then

$$\left| f * g_\delta(x) - f(x) \right| \leq \int_{x-\delta}^{x+\delta} | f(y) - f(x) |\, g_\delta(x-y)\, dy$$

$$\leq \int_{x-\delta}^{x+\delta} \varepsilon\, g_\delta(x-y)\, dy = \varepsilon$$

$\Rightarrow \left| f * g_\delta(x) - f(x) \right| < \varepsilon$

for $x \in [a,b]$ if $\delta < \delta_0$.

Properties of ~~the~~ convolutions

- $f * g(x) = g * f(x)$

- $(f * g) * h(x) = f * (g * h)(x)$

- $(f+g) * h(x) = f * h(x) + g * h(x)$

- The function $f * g$ has properties of f and g.

Suppose g is diff'l, then $f * g$ is diff'l.

Consider : $$\frac{f * g(x+h) - f * g(x)}{h}$$

$$= \int_{-R}^{R} \frac{f(y)\left[g(x+h-y) - g(x-y)\right]}{h} dy$$

If $g \in C'$ then

$$\lim_{h \to 0} \underbrace{\frac{g(x+h-y) - g(x-y)}{h}}_{\left(\frac{\Delta_h g(x-y)}{h}\right)} = g'(x-y)$$ and the

convergence is the ~~a~~ limit over any bounded interval.

$$\Rightarrow \frac{d}{dx} f * g = \lim_{h \to 0} \frac{\Delta_h f * g(x)}{h} = \int_{-R}^{R} f(y) \cdot g'(x-y)\, dy.$$

$$= f * g'(x)$$

$$x \cdot 1 = x \quad \forall x$$

Q: Does there exist a function $\nexists$ i s.t.

$$f * i = f \qquad \forall f \in C^0.$$

A: NO!

$i(x)$ would have to be non-zero on a set which is big enough and

$$\int i(x-y)\,dy = 1$$

and:

$$\int f(y) i(x-y)\,dy = f(x) \qquad \forall x, \ \forall f \in C^0$$

—— which cannot be true.

Suppose $g_n(x)$ is a seq. of R-I. functions s.t.:

1. $g_n(x) \geq 0$
2. $\int g_n(x)\,dx = 1 \quad \forall n$
3. Given ~~$\varepsilon > 0$~~ $\varepsilon, \delta > 0$ $\exists N$ s.t. if $n \geq N$ then $\displaystyle\int_{|x| > \delta} g_n(x)\,dx < \varepsilon$

A seq. of functions with these properties is called an approximate identity

THEOREM : If $\langle g_n \rangle$ is an approximate identity and f is a continuous function, which is zero outside $[a, b]$ ~~then $f *$~~ for some choice of a, b, then

$f * g_n$ converges uniformly to f as $n \to \infty$.