09/05/17

MATH 508 - Lecture 3

AXIOM (Principle of Mathematical Induction)

Let $A \subseteq \mathbb{N}$

1. $1 \in A$
2. If $n \in A$, then $n+1 \in A$

then $A = \mathbb{N}$

Claim:

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

Pf:

$n = 1$ (base case)

$$1 = \frac{1(1+1)}{2} = 1 \quad \checkmark$$

Assume inductively that the formula is true for n. We want to prove the formula for $(n+1)$.

$$\text{LHS} = \sum_{j=1}^{n+1} j = \sum_{j=1}^{n} j + (n+1)$$

$$= \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{(n+1)(n+2)}{2} = \text{RHS}$$

A and B are sets

A function $f: A \longrightarrow B$ is the assignment of a unique element of B to each element of A, denoted $f(a)$.

We say that f is one-to-one (injective) if $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

We say that f is onto (surjective) if $\forall\, b \in B \;\; \exists\, a \in A$ with $f(a) = b$

$$\underline{\text{Im}}\, A = \{ f(a) : a \in A \}$$

f is surjective if $f(A) = B$.

A map that is both injective and surjective is called <u>bijective</u>.

<u>Proposition</u>: ~~If~~ $f: A \to B$ is bijective iff there exists a map $g: B \to A$ such that

$$f \circ g = Id_B \quad \text{and} \quad g \circ f = Id_A$$

<u>Proof</u>:

Counting [Cardinality of a set]

$|A| \stackrel{d}{=}$ cardinality of the set A

$$|\emptyset| = 0$$

We say that a set A has cardinality n if there is a bijective map

$$f : A \to \{1, 2, \dots, n\}$$

How do we know that there ~~exists~~ isn't also a bijective map:

$$f' : A \to \{1, \dots, m\} \quad \text{where } m \neq n$$

Proposition. If $f : \{1, \dots, n\} \to \{1, \dots, m\}$, $m, n \in \mathbb{N}$ is a bijection, then $n = m$

Proof: We prove the proposition by inducting on n.

P_n: If $f : \{1, \dots, n\} \to \{1, \dots, m\}$ is a bijection then $m = n$

P_1 is true $\quad f : \{1\} \to \{1, \dots m\}$

$$f(1) = m = 1$$

Assume that P_n is true.

let $j = f(n+1) \in \{1, \dots m\}$

$$\{1, m\} = \{1, \dots j-1\} \cup \{j\} \cup \{j+1, \dots m\}$$

$$f : \{1, \dots, n\} \longrightarrow \{1, \dots j-1\} \cup \{j+1, \dots m\}$$

$$g : \{1 \dots j-1\} \cup \{j+1, \dots m\} \longrightarrow \{1, \dots n\}$$

$$\tilde{f} = f\big|_{\{1, \dots n\}}$$

$$g \circ \tilde{f} : \{1, \dots, n\} \longrightarrow \{1, \dots m-1\}$$

which is a bijection

By the induction hypothesis,

$$m - 1 = n$$

$$\Rightarrow \boxed{m = n+1}$$

which completes the induction.

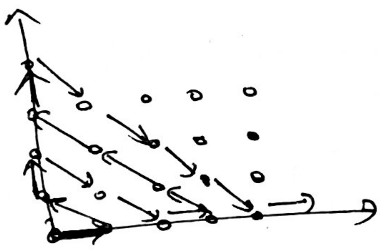
A set for which $\exists$ a bijection $f: A \to \{1, \ldots, n\}$ is a finite set with $|A| = n$.

If no such bijection exists then $|A| = \infty$ or A is an infinite set

If $\exists$ a bijection $f: \mathbb{N} \to A$, then we say that A is a countable set.

1. $\mathbb{Z} = \mathbb{N}$

2. $\mathbb{N} \times \mathbb{N} = \{(n,m) : n, m \in \mathbb{N}\}$



show $\mathbb{N} \times \mathbb{N}$ is a countably infinite set.

A is countable $\Rightarrow \exists\, f: \mathbb{N} \to A$ which is bijective

$A = \{a_1, a_2, \ldots\}$

$a_j = f(j)$

<u>Proposition</u> : Suppose A is countably infinite and $B \subseteq A$, then B is either finite or countably infinite.

<u>Pf</u>:

Since A is countable we can enumerate it

$$A = \{a_1, a_2, a_3, \dots\}$$

[If $C \subseteq \mathbb{N}$, then C has a smallest element]

$$f(1) = \inf\{j : a_j \in B\}$$

$$f(2) = \inf\{j : a_j \in B \setminus \{f(1)\}\}$$

Recursively, we have

$$f(k) = \inf\{j : a_j \in B \setminus \{a_{f(1)}, \dots a_{f(k-1)}\}$$

unless $B \setminus \{a_{f(1)} \dots a_{f(k-1)}\}$ is ϕ

Either $\exists\ k$ s.t. $B = \{a_{f(1)}, \dots, a_{f(k-1)}\}$ or <u>not</u>.

If $B = \{a_{f(1)}, \dots a_{f(k-1)}\}$ then

$$|B| = k-1$$

otherwise $f: \mathbb{N} \longrightarrow \mathbb{N}$ which is injective

$f(j) < f(j+1)$

... (complete the proof!)

Gives the basic ideas to define sizes of infinite sets.

① If $B \subseteq A$, then $|B| \leq |A|$

② If A and B are sets s.t. $\exists$ a bijection $f: A \to B$, then $|A| = |B|$

Schröder Bernstein Theorem :

If A and B are sets such that $\exists$ injective maps $f: A \to B$ and $g: B \to A$, then $\exists$ a bijective map $h: A \to B$

Let $2^{\mathbb{N}}$ = all infinite seq. of 0_s & $1s$

<u>Th</u>. $2^{\mathbb{N}}$ is not countable.

<u>Pf</u>: Suppose not.

Let $\{\beta^{(1)}, \beta^{(2)}, \dots\}$ be the enumeration of $2^{\mathbb{N}}$

Construct a new sequence

$$\alpha_j = \begin{cases} 0 & \text{if } \beta_j^{(j)} = 1 \\ 1 & \text{if } \beta_j^{(j)} = 0 \end{cases}$$

$\alpha \notin \{\beta^{(1)}, \beta^{(2)}, \dots\}$

(complete... !)