MATH 508 - Lecture 4

Proposition: $(A \neq \phi)$
If $A \subseteq \mathbb{N}$, then A has a smallest element.
(Well-ordering principle)

Proof:

Assuming the finite case.

$\exists$ an integer $n_0 \in \mathbb{N}$ s.t $A \cap \{1, \ldots, n_0\} \neq \phi$

$$\mathbb{N} = \bigcup_{n=1}^{\infty} \{1, \ldots, n\}, \quad \text{otherwise } A = \phi$$

$$|A \cap n_0| < \infty$$

Hence $\exists$ $m \in A \cap \{1, \ldots, n_0\}$ which is the smallest element of this set.

$$m \leq n_0 \quad \text{and} \quad n_0 < \text{ all } x \in \{n_0+1, \ldots\}$$

$$A = A \cap \{1, \ldots, n_0\} \cup A \cap \{n_0+1, \ldots\}$$

$\cancel{m \leq}$ $\Big\{$ $m \leq$ all elements m $\{1, \ldots, n_0\}$
and $\{n_0+1, \ldots\}$

$\Rightarrow$ m is the smallest element of A. $\blacksquare$

$\mathbb{Q}$: rational numbers are defined as:

$\frac{m}{n}$ where $m \in \mathbb{Z}$, $n \in \mathbb{Z} \setminus \{0\}$

$\frac{m}{n} \sim \frac{p}{q}$ iff $mq = np$

$\frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}$

$\frac{m}{n} \cdot \frac{p}{q} = \frac{m \cdot p}{n \cdot q}$

$\rightarrow$ $\mathbb{Q}$ is an ordered field.

~~Arithme~~

$\frac{m}{n} < \frac{p}{q}$ iff $mq < pn$

$\forall r, q \in \mathbb{Q}$

$r < q$ xor $r = q$ xor $r > q$

order relation is compatible with arithmetic

If $r < p$ then

$r + p < p + q \ \forall q \in \mathbb{Q}$

$r \cdot q < p \cdot q \ \forall q \in \mathbb{Q}$

$r < p$ iff $-r > -p$

Absolute value ~~of~~

$$|r| = \begin{cases} r & \text{if } r \geq 0 \\ -r & \text{if } r < 0 \end{cases}$$

(a) $0 \leq |r|$ and $|r|$ iff $r = 0$.

(b) $|r+q| \leq |r| + |q|$ (Triangle inequality)

(c) $-q \leq r \leq q$ iff $|r| \leq |q|$

(d) $|xy| = |x| \cdot |y|$

Define : $d(r,q) = |r-q|$

(1) $d(r,q) = 0$ iff $r = q$
(2) $d(r,q) = d(q,r)$
(3) $d(r,q) \leq d(r,s) + d(s,q)$

Define : powers

$x^1 \stackrel{d}{=} x, \quad \forall x \in \mathbb{Q}$

$x^n \stackrel{d}{=} x^{n-1} \cdot x, \quad \forall n \in \mathbb{N}$

$x^0 \stackrel{d}{=} 1, \quad \forall x \in \mathbb{Q}$

$x^n \cdot x^m = x^{n+m}$

$(x^n)^m = x^{nm}$

$(xy)^n = x^n \cdot y^n$

$x^n = 0 \Leftrightarrow x = 0, \quad n \in \mathbb{N}$

$x, y \in \mathbb{Q}$

$x \cdot y = 0$ iff $x = 0$ or $y = 0$

$0 \leq y \leq x$ then

$0 \leq y^n \leq x^n \quad \forall n \in \mathbb{N}$

$|x^n| = |x|^n$

Proposition : $\mathbb{Q}$ has the Archimedean property.

If r, s are positive rational numbers, $r \neq s$, then there is a +ve int, N such that $N \cdot r > s$

Proof :

let $r = \frac{m}{n}$, $s = \frac{p}{q}$

where m, n, p, q are all positive

$$N = n \cdot p + 1$$

$$N \cdot r = p \cdot m + \frac{m}{n} > p \geq \frac{p}{q}$$

$\mathbb{R}$ (is an {ordered field.}), is a set with operations $+, \cdot$

(A1) $(a+b)+c = a+(b+c)$

(A2) $a+b = b+a$

(A3) $\exists\ 0 \in \mathbb{R}$, s.t. $a = 0 = a$

(A4) $\forall a \in \mathbb{R}$, $a + (-a) = 0$

(M1) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(M2) $a \cdot b = b \cdot a$

(M3) $\exists\ 1 \neq 0$ s.t. $1 \cdot a = a \cdot 1 = a$ $\forall a \in \mathbb{R}$

(M4) $\forall a \neq 0$ $\exists\ a^{-1}$ s.t. $a \cdot a^{-1} = 1$

(D) $a \cdot (b+c) = a \cdot b + a \cdot c$

(01) $\forall\ a, b \in \mathbb{R}$;

$a < b$ xor $a = b$ xor $a > b$

(02) if $a < b$, $b < c \Rightarrow a < c$

(03) $a < b \Rightarrow a + c < b + c$

(04) $a < b$, $c > 0 \Rightarrow a \cdot c < b \cdot c$

(05) If $0 < a, b$ then $\exists$ int $n \in \mathbb{N}$ s.t.

$b < n \cdot a$.

Let $A \subseteq \mathbb{R}$

We say that A is bounded from above if there is a number $c \in \mathbb{R}$ such that $\forall\ x \in A$, $x \leq c$.
c is called an upper bound for A.

A number b is called a "least upper bound" for A if $\forall\ x \in A$, $x \leq b$ and if c is any other upper bound for A, then $b \leq c$.

(06) Every set A that is bounded from above has a least upper bound.

$\left[(06) \Rightarrow (06') \right]$ greatest upper bound property.

Note: $\mathbb{Q}$ does not satisfy (06)