09/14/17

MATH 508 - Lecture 6

Thm : If $a > 0$, $n \in \mathbb{N}$, then there is a unique $x > 0$, such that $x^n = a$.

Pf :

(Uniqueness)

If $x, y \in \mathbb{R}$ $x^n - y^2 = (x-y)(x^{n-1} + x^{n-2}y + \cdots + \cdots y^{n-1})$

$$= (x-y)\underbrace{\left(\sum_{j=0}^{n-1} x^{n-j-1} \cdot y^{j}\right)}_{(>0)}$$

$x, y > 0$ then

$x^n > y^n$ iff $x > 0$

Only consider the case that $a > 1$.

[If $a < 1$, then $b = \frac{1}{a}$, $b > 1$

then $y^n = b$ then

$(y^{-1})^n = b^{-1} = a$]

$a > 1$:

$$A_n = \{x \in \mathbb{R} : x^n < a\}$$

A_n is not empty, $1 \in A_n$

$\boxed{\text{Fact} : a < a^n \ ; \ a > 1}$

$x^n < a$

So A_n is a set bounded from above. There is a l.u.b. of $A_n = b$. [$b > 0$, as $a > 0$]

Claim : $b^n = a$

Pf: If $b^n \neq a$, either $b^n < a$ or $b^n > a$

Idea:

Case 1 : $b^n < a$

we will show $\exists$ $b < c$ and $c^n < a$

so b is not an upper bound for A_n

Case 2 : $b^n > a$

we will show $\exists$ $c < b$ s.t. $c^n > a$

so b is not the l.u.b. for A_n.

(1) $b^n < a$

$$c^n - b^n = (c-b)(c^{n-1} + \dots + b^{n-1})$$

let $c < b+1$

$$\Rightarrow c^n - b^n < (c-b) \cdot n \cdot (b+1)^{n-1} \quad \text{--- (1)}$$

want $c^n < a$

$$c^n = c^n - b^n + b^n < a$$

$$\Rightarrow \quad c^n - b^n < a - b^n \quad \text{——— } (2)$$

$$c^n - b^n < n\cdot(c-b)(b+1)^{n-1} \quad \text{——— } (3)$$

$\{$from (1)$\}$

set $a - b^n = n(c-b)(b+1)^{n-1}$

$$\boxed{c = \frac{a-b^n}{n\cdot(b+1)^{n-1}} + b}$$

Clearly $\boxed{c > b}$ $\left[\begin{array}{l} a-b^n > 0 \\ n\cdot(b+1)^{n-1} > 0 \end{array}\right]$

By choice of c, we have:

$c^n - b^n < a - b^n$ $\left[n(c-b)(b+1)^{n-1} = a - b^n \right]$

$$\Rightarrow \boxed{c^n < a}$$

$\{$ Note: If c was greater than $b+1$, setting $c = b+1$ would work $\}$

~~Case~~ 2 $\underline{b^n > a}$

need to show $\exists$ $c \cancel{<} b$ such that

$c^n > a$

$$b^n - c^n < b^n - a \qquad \text{——— (4)}$$

$$b^n - c^n = (b-c)\left(\sum_{j=0}^{n-1} b^{n-1-j} \cdot c^j\right)$$

$$< (b-c)\, n \cdot b^{n-1} \qquad \{b > c\}$$

$$\cancel{(b \cancel{<} c)} \qquad \text{——— (5)}$$

Using (4) and (5)

set $b^n - a = (b-c)\, n \cdot b^{n-1}$

$$\boxed{c = b - \frac{b^n - a}{n \cdot b^{n-1}}}$$

which means that b is not the l.u.b.

$$\Rightarrow \boxed{b^n = a}$$

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CONCLUDE :

$\forall\ a \in (0, \infty),\ n \in \mathbb{N}$

there is a unique x such that

$x^n = a\ ,\ x > 0.$

Defn :

A sequence of real numbers is a map $f : \mathbb{N} \longrightarrow \mathbb{R}$.

- You can have sequence of other "stuff" as well [points in $\mathbb{R}^2$, etc.]
- A sequence is not a set — it can have repeating terms.
- Terms ~~come~~ are ordered
- denoted as $a_1, a_2, \ldots a_n, \ldots$ —OR— $\langle a_n \rangle$

Defn

A sequence converges to x^* iff given $N \in \mathbb{N}$, there is an $M \in \mathbb{N}$ such that for all $n \geq M$,

$$|x_n - x^*| \leq \frac{1}{N},$$ and we write:

$$\boxed{\lim_{n \to \infty} x_n = x^*}$$

Prop. If $\langle x_n \rangle$ has a limit, then the limit is unique.

Pf: Suppose not.

Then $\exists$ $x_1^* \neq x_2^*$ such that

$$\lim_{n \to \infty} x_n = x_i^* \qquad i = 1, 2$$

$$0 \neq |x_1^* - x_2^*| > \frac{1}{N} \qquad \text{(else we're done)}$$

$\exists$ M_1 s.t. $|x_n - x_1^*| < \frac{1}{2N}$ if $n \geq M_1$

M_2 s.t. $|x_n - x_2^*| < \frac{1}{2N}$ if $n \geq M_2$

$$|x_1^* - x_2^*| \leq |x_1^* - x_n| + |x_n - x_2^*| \qquad \text{[Triangle ineq.]}$$

If $n \geq \max\{M_1, M_2\}$

$$< \frac{1}{2N} + \frac{1}{2N} = \frac{1}{N}, \quad \text{a contradiction}$$

$\Rightarrow$ the limit is unique

$\blacksquare$

Main Problem of Analysis :

Decide when a sequence has a limit

Defn : A sequence $\langle x_n \rangle$ is monotone increasing if $x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n \dots$

A sequence is montone decreasing if

$$x_1 \geq x_2 \geq \dots \geq x_n \geq \dots$$

Theorem : A monotone increasing sequence has a limit iff it is bounded.

Proof :

($\Rightarrow$) Any convergent sequence is bounded

Obs : Any finite set is bounded

Because $\langle x_n \rangle$ converges to x^*, $\exists M$ such that

$$|x_n - x^*| \leq 1 \quad \text{for } n \geq M$$

Using the triangle ineq,

$$|x_n| \leq |x^*| + |x_n - x^*| \leq |x^*| + 1 \quad \text{if } n \geq M$$

If we set $m = \max\{|x_1|, |x_2|, \dots |x_{M-1}|, 1 + |x^*|\}$

then $|x_n| \leq m$ for all $n \in \mathbb{N}$.

($\Leftarrow$) A monotone increasing bounded sequence has a limit.

let $x^* = \text{l.u.b} \{ x_1, x_2, \dots x_n \}$

<u>Claim</u> : $\lim\limits_{n \to \infty} x_n = x^*$

we know : $x^* - x_n \geq 0$ [x^* is an upper bd.]

want to show : for a given $N \in \mathbb{N}$, $\exists M \in \mathbb{N}$ such that $x^* - x_n \leq \frac{1}{N}$ if $n \geq M$

$x^* - x_n \geq x^* - x_{n-1} \cdots \geq 0$ [Note this is a decreasing sequence]

If this condition does not hold for some N, then

$x^* - x_n > \frac{1}{N} \quad \forall n$ [Because the sequence $\langle x^* - x_n \rangle$ is decreasing]

$x^* - \frac{1}{N} > x_n \quad \forall n$

$\Rightarrow \left(x^* - \frac{1}{N} \right)$ is an upper bound for $\langle x_n \rangle$

which is a contradiction

(x^* was the l.u.b.)

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