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MATH 508 – Lecture 7

- $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.

  The simplest such numbers (in some sense) are called algebraic numbers.

  Let $p \in \mathbb{Q}[x]$ [notation in PSet 1, Q.6]

$$p(x) = a_n x^n + \dots + a_0$$

$$a_n \neq 0$$

$$= a_n \underbrace{\left[x^n + \frac{a_{n-1}}{a_n} x^{n-1} + \dots + \frac{a_0}{a_n} \right]}_{\mathbb{Q}[x]}$$

monic polynomial ($a_n = 1$)

~~A polynomial~~

- A number $\alpha \in \mathbb{R}$ is algebraic if $\exists\ p \in \mathbb{Q}[x]$ such that $p(\alpha) = 0$.

- $\mathbb{Q}[x]$ is countable [proved in PSet 1]

- $R_p = \{ \alpha \mid p(\alpha) = 0 \}$

  all algebraic numbers must appear in $\bigcup_{p \in \mathbb{Q}[x]} R_p$; which implies the set of algebraic numbers are countable.

- Irrational Numbers that are not algebraic are called <u>transcendental</u> numbers.

<u>Note</u>: Rationals are countable, algebraic #s are countable $\Rightarrow$ <u>transcendental #s are uncountable.</u>

<u>Thm</u>: Suppose that α is algebraic of degree $k \geq 2$ [-OR- that α is not rational], then there is a constant K_1 such that for all rational numbers ~~p/q~~ $r = p/q$ in lowest terms

$$|\alpha - r| \geq \frac{K_1}{q^k}$$

<u>Pf</u>:

Let $p \in Q[x]$ be a monic polynomial of degree $k \geq 2$ such that $p(\alpha) = 0$.

If $|\alpha - r| \geq 1$, then its easy to see that

$|\alpha - r| \geq \frac{K_1}{q^k}$ for some K_1

Hence we assume,

$|\alpha - r| < \varepsilon$, $\varepsilon > 0$; and $|\alpha - r| < 1$

By the mean value th,

$$P(r) = P(r) - P(\alpha) = P'(s)(r - \alpha)$$

where $s \in (r, \alpha)$ [or (α, r)]

$|p'(s)| \leq L$ for $s \in [\alpha-1, \alpha+1]$

$|P(r)| \leq L|r-\alpha|$

$$\boxed{|r-\alpha| \geq \frac{1}{L} P(r)} \quad \text{—— } 1$$

can write : $P(r) = P(P/q) = \sum_{j=0}^{k} a_j \left(\frac{p}{q}\right)^j$

$$= \sum_{j=0}^{k} \frac{n_j}{m_j} \left(\frac{p}{q}\right)^j$$

$$= \frac{1}{M} \cdot \frac{1}{q^k} \sum_{j=0}^{k} n_j \hat{m}_j p^j q^{k-j}$$

$$M = m_0 \cdot m_1 \cdots m_k$$

$$\hat{m}_j = m_0 \cdots m_{j-1} \cdot m_{j+1} \cdots m_k$$

$\exists\ \varepsilon > 0$, s.t.

$P(x) \neq 0$ for $x \in (\alpha-\varepsilon, \alpha+\varepsilon) \setminus \{\alpha\}$

If $r \in [\alpha - \varepsilon/2, \alpha + \varepsilon/2]$, then

$P(r) \neq 0$ —— use this in 1

$$\Rightarrow \left|\frac{p}{q} - \alpha\right| \geq \frac{1}{L} \cdot \frac{1}{Mq^k} \qquad \left\{ \text{Here } \frac{1}{LM} = K_1 \right\}$$

$$\left\{ \left| \sum_{j=0}^{k} n_j \hat{m}_j p^j q^{k-j} \right| \geq 1 \right\}$$

Example :

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{10^{k!}} \qquad k! = 1.2.3 \ldots k$$

$$\alpha_N = \sum_{k=1}^{N} \frac{1}{10^{k!}} = \frac{P_N}{10^{k!}} \qquad \{ P_N \in \mathbb{Z} \}$$

$$|\alpha - \alpha_N| = \sum_{k=N+1}^{\infty} \frac{1}{10^{k!}}$$

Claim : $|\alpha - \alpha_N| < \dfrac{2}{10^{(N+1)!}}$

$\Rightarrow$ α is not algebraic.

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Claim : e is not a rational number.

Proof : Suppose not. Then $e = p/q$

Let $S_q := \sum_{n=0}^{q} \frac{1}{n!}$

Then we can show :

$$|e - S_q| = \sum_{n=q+1}^{\infty} \frac{1}{n!} \leqslant \frac{1}{q!\,q}$$

$$\Rightarrow \boxed{q!\,|e - S_q| \leq \frac{1}{q}}$$

Now consider,

$$q!\left(\frac{p}{q} - \sum_{n=0}^{q} \frac{1}{n!}\right)$$

$$= \underbrace{\left(p(q-1)! - \sum_{n=0}^{q} \frac{q!}{n!}\right)}_{\text{cannot be zero, since we have thrown away a few terms}}$$

term should $\in \mathbb{Z}$.

$$\Rightarrow \boxed{q!\,|e - S_q| \geq 1}$$, a contradiction.

$\blacksquare$

- Diophantine approximations properties:

how well can a real number be approximated by a rational number.

- Decimal Expansions

Let $a \in \mathbb{R}$,

$\exists$ $a_0 \in \mathbb{Z}$ such that:

$$a_0 < a \leq a_0 + 1$$

$$\frac{a_1}{10} < a - a_0 \leq \frac{a_1}{10} + \frac{1}{10}$$

where $a_1 \in \{0, 1, \dots 9\}$

$(0,1) = (0, \frac{1}{10}] \cup (\frac{1}{10}, \frac{2}{10}] \cup \dots$
$\cup \dots (\frac{9}{10}, 1]$

can recursively split up the above interval

$(0, \frac{1}{10}) = (0, \frac{1}{10^2}] \cup \dots$
$\dots \cup (\frac{9}{10^2}, \frac{1}{10}]$

Then we can also have a_2 s.t.

$$\frac{a_2}{10^2} < a - \left(a_0 + \frac{a_1}{10}\right) \leq \frac{a_2}{10^2} + \frac{1}{10^2}$$

$$a_2 \in \{0, 1, \dots 9\}$$

Continuing this way:

$\exists$ $a_1, a_2, \dots a_j, a_{j+1} \in \{0, 1, \dots 9\}$

$$\frac{a_{j+1}}{10^{j+1}} < a - \left(a_0 + \frac{a_1}{10^1} + \dots \frac{a_j}{10^j}\right) \leq \frac{a_{j+1}}{10^{j+1}} + \frac{1}{10^{j+1}}$$

let $x_j = a_0 + \frac{a_1}{10} + \cdots \frac{a_j}{10^j}$

which is increasing

(1) $x_j < a \quad \forall j$

(2) $a - x_j < \frac{1}{10^j}$

$(a - x_j)$ is a decreasing sequence

$$\boxed{\lim_{j \to \infty} x_j = a} \qquad \left(\frac{1}{10^j} \leq \frac{1}{j} \right)$$

We can do the above analysis for any base.
In particular, $a = a_0 + \sum_{j=1}^{\infty} \frac{b_j}{2^j}$; $b_j \in \{0, 1\}$.

Can you do arithmetic with two real numbers that have infinite decimal expansions?

Thm : $\mathbb{R}$ is an uncountable set.

Pf : we will prove that the interval $[0,1)$ is uncountable.

$$\mathbb{R} = \bigcup_{n=-\infty}^{\infty} [n, n+1)$$

If even one of these intervals is not countable, $\Rightarrow$ that $\mathbb{R}$ is not countable.

$2^{\mathbb{N}}$ = the collection of all binary sequences is ~~countable~~ uncountable

$$f : 2^{\mathbb{N}} \to [0,1]$$

$$\{b_1, b_2, \dots\} \longrightarrow \sum_{j=1}^{\infty} \frac{b_j}{2^j}$$

If $x_n = \sum_{j=1}^{n} \frac{b_j}{2^j}$; $x_n \leq x_{n+1} \leq \dots$

$$x_n = 1 - \frac{1}{2^n} \leq 1$$

Let $\beta = (b_1, b_2, \dots)$

$$\sigma(\beta) = \sum_{j=1}^{\infty} \frac{b_j}{2^j}$$

Q. When is it possible for 2 binary sequences $\beta^{(1)}, \beta^{(2)}$ to have $\sigma(\beta^{(1)}) = \sigma(\beta^{(2)})$?

A. pairs of the form :

$$\underbrace{0000\ldots 00}_{k}1111\ldots$$

$$\underbrace{0000\ldots 0}_{k-1}100\ldots$$

The first $(k-1)$ positions can have any vector $\in \{0,1\}^{k-1}$

represent the same ~~bi~~ number.

Claim (*): These are the only pairs such that
$$\sigma(\beta^{(1)}) = \sigma(\beta^{(2)})$$

This set $\mathcal{E}$ is countable. [the set of all numbers that have two representations]

$\sigma : 2^{\mathbb{N}} \setminus \mathcal{E} \to [0,1)$ in an 1-1 way.

$\sigma(2^{\mathbb{N}} \setminus \mathcal{E})$ is a subset of $[0,1)$ with the same cardinality as $(2^{\mathbb{N}} \setminus \mathcal{E})$

$2^{\mathbb{N}} \setminus \mathcal{E}$ is also uncountable, and therefore so is $[0,1)$

Proof of Claim (*):

$$\sigma(\beta^{(1)}) = \sigma(\beta^{(2)})$$

$$\beta^{(1)} = b_1^{(1)}, b_2^{(1)}, \ldots$$
$$\beta^{(2)} = b_1^{(2)}, b_2^{(2)}, \ldots$$

$\exists$ k such that:

$$b_j^{(1)} = b_j^{(2)} \quad 1 \leq j \leq k$$
$$b_{k+1}^{(1)} = 0 \qquad b_{k+1}^{(2)} = 1$$

$$\sum_{j=k+2}^{\infty} \frac{b_j^{(1)}}{2^j} \leq \frac{1}{2^{k+1}}$$; the inequality is strict if $b_j^{(1)} \neq 1$ for some $j > k+1$

$$\sum_{j=k+1}^{\infty} \frac{b_j^{(2)}}{2^j} \geq \frac{1}{2^{k+1}}$$; the inequality is strict if $b_j^{(2)} \neq 0$ for some $j \geq k+2$.

So the only way $\sigma(\beta^{(1)}) = \sigma(\beta^{(2)})$ if

$b_j^{(1)} = 1$ for all $j > k+1$

and $b_j^{(2)} = 0$ for all $j > k+1$

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