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# MATH 508 - Lecture 9

## RECAP

$\langle x_n \rangle$ is a bounded sequence

$$y_j = \sup\{x_j, x_{j+1}, \dots\}$$

$$y_j \geq y_{j+1} \geq \dots \quad ; \quad y^* = \lim_{n\to\infty} y_j \text{ exists}$$

Want to prove $\exists$ a subsequence $\langle x_{n_j} \rangle$

s.t. $\displaystyle\lim_{j\to\infty} x_{n_j} = y^* = \lim_{j\to\infty} \sup x_j$

$$\{ \cancel{y}j : \exists\, n_j \text{ with } n_j \geq j \text{ and } y_j = x_{n_j} \} =: \mathcal{E}$$

Either $|\mathcal{E}| = \infty$ or $\mathcal{E}$ is finite.

Case 1 : $|\mathcal{E}| = \infty$

(proof in lecture 8)

Case 2 : $\mathcal{E}$ is finite

$j_0$ is the largest index in $\mathcal{E}$

let $j_1 = j_0 + 1$

$\nexists$ $j_1 \leq n$ such that $x_n = y_{j_1}$

$\forall j > j_0$ $\nexists$ $x_n = y_j$ , $n \geq j$

In this case given any $\# l$

the set $\{k \geq j : x_k > y_j - \frac{1}{l}\}$ is infinite

$j_1 = j_0 + 1$

let $l = 1$

$\{j_1 \leq k : x_k > y_{j_1} - \frac{1}{l}\}$

Choose one of the $k$'s ; $k_1 \geq j_1$

$x_{k_1} \geq y_{j_1} - \frac{1}{l}$

Choose $j_2 > k_1$ , $\{k : x_k > y_{j_2} - \frac{1}{2}\}$

choose $k_2 \geq j_2$ from this set

$\vdots$

We've chosen $j_1 \leq k_1 < j_2 \leq k_2 < \dots < j_l \leq k_l$

s.t. $x_{k_i} \geq y_{j_i} - \frac{1}{i}$ $\quad i = 1, \dots, l$

Choose $j_{l+1} > k_l$ and consider

$$\left\{ k \geq j_{l+1} : x_k > y_{j_{l+1}} - \frac{1}{l+1} \right\}$$

Choose $k_{l+1} \geq j_{l+1}$ such that:

$$x_{k_{l+1}} > y_{j_{l+1}} - \frac{1}{l+1}$$

$$\boxed{y_{j_l} - \frac{1}{l} < x_l < y_{j_l}}$$

$$\underset{y^*}{\downarrow} \qquad\qquad\qquad\qquad \underset{y^*}{\downarrow}$$

By the <u>Squeeze theorem</u>,

$$\lim_{l \to \infty} x_{k_l} = y^*$$

Hence there exists a subsequence $\langle x_{k_l} \rangle$ such that:

$$\boxed{\lim_{l \to \infty} x_{k_l} = \limsup_{n \to \infty} x_n}$$

Squeeze Thm : If $x_n \le y_n \le z_n$ are sequences

such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x^* \quad \text{then}$$

$\lim_{n\to\infty} y_n$ exists and equals to $x^*$.

Pf :

$$0 \le y_n - x_n \le z_n - x_n$$

given $N$, $\exists$ $M$ s.t.

$$|y_n - x_n| \le |z_n - x_n| \le \frac{1}{N} \quad \forall\, n \ge M$$

and therefore $\lim_{n\to\infty} |y_n - x_n| = 0$

$$y_n = x_n + (y_n - x_n)$$

Applying limit laws

~~$\lim_{n\to\infty} = \lim$~~

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n + \cancelto{0}{\lim_{n\to\infty} (y_n - x_n)}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} y_n = x^*}$$

$\langle x_n \rangle$ is a bounded seq.

$$y_j = \sup\{x_j, x_{j+1}, \cdots\}$$
$$z_j = \inf\{x_j, x_{j+1}, \cdots\}$$

$$z_j \leq x_j \leq y_j$$

We've proved that $\lim\limits_{j\to\infty} y_j = y^*$ is a limit point of the sequence $\langle x_n \rangle$

(1) Any bounded seq. has limit points $\iff$ $\exists$ a convergent subsequence

**Thm**: Suppose that $x^*$ is a limit point of $\langle x_n \rangle$, then

$$\liminf_{j\to\infty} x_j \leq x^* \leq \limsup_{j\to\infty} x_j$$

Pf: If $x^*$ is a limit pt, then $\exists$ a subseq. $\langle x_{n_j} \rangle$ s.t.

$$\lim_{j\to\infty} x_{n_j} = x^*$$

Also $z_{n_j} \leq x_{n_j} \leq y_{n_j}$

$$x_{n_j} - z_{n_j} \geq 0 \quad \forall j$$

$$0 \leq \lim_{j\to\infty} (x_{n_j} - z_{n_j})$$

$$= x^* - \liminf_{j\to\infty} x_j$$

$$x^* \geq \liminf_{j\to\infty} x_j$$

$|||^{ly}$, $\quad x^* \leq \limsup_{j\to\infty} x_j$

$\blacksquare$

__Bolzano - Weierstrass Thm__ :

If $\langle x_n \rangle$ is a bounded seq. of reals then $\langle x_n \rangle$ has a convergent subsequence

__Thm__ : $\langle x_n \rangle$ is convergent iff

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$$

$(\Leftarrow)$ let $\liminf x_n = \limsup x_n = x^*$

$$z_n \leq x_n \leq y_n \quad \forall n$$

Squeeze Thm shows:

$$\lim_{n \to \infty} x_n = x^*$$

$(\Rightarrow)$ If $\lim_{n \to \infty} x_n = x^*$

Then given $l$, $\exists$ $M_l$

s.t. $x^* - \frac{1}{l} \leq x_n \leq x^* + \frac{1}{l} \Leftrightarrow |x_n - x^*| \leq \frac{1}{l}$

$\forall n \geq M_l$

$$x^* - \frac{1}{l} \leq y_j \leq x^* + \frac{1}{l} \quad \text{if } j \geq M_l$$

$$x^* - \frac{1}{l} \leq z_j \leq x^* + \frac{1}{l} \quad \text{if } j \geq M_l$$

Choose $j_l \geq M_l$

$$x^* - \frac{1}{l} \leq y_{j_l} \leq x^* + \frac{1}{l}$$

$M_{l+1}$ s.t. $|x_n - x^*| \leq \frac{1}{l+1}$ $\quad \forall n \geq M_{l+1}$

$$j_{l+1} > \max\{M_{l+1}, j_l\}$$

$$x^* - \frac{1}{l+1} \leq y_{j_{l+1}} \leq x^* + \frac{1}{l+1}$$

$$\Rightarrow \boxed{\lim_{l \to \infty} y_{j_l} = x^*}$$

$$\lim_{n \to \infty} x_n = x^*$$

$$|x_n - x_m| \leq |x_n - x^*| + |x^* - x_m|$$

Given $N$, $\exists M$ s.t. $|x_n - x^*| \leq \frac{1}{N}$

$$|x_m - x^*| \leq \frac{1}{N} \quad \text{if } n, m \geq M$$

then $\boxed{|x_n - x_m| \leq \frac{2}{N} \quad \forall\, n, m \geq M.}$ ☆

Defn:

A sequence $\langle x_n \rangle$ is called a Cauchy sequence if given $N$, $\exists M$ s.t. $|x_n - x_m| \leq \frac{1}{N}$ if $n, m \geq M$.

☆ A convergent sequence is a Cauchy sequence.

Thm : A Cauchy sequence is convergent

Pf : Fix an $N \in \mathbb{N}$, $\exists\, M_N$ s.t.

$$|x_n - x_m| \leq \frac{1}{N} \quad \text{if } n, m \geq M_N$$

$$x_{M_N} - \frac{1}{N} \leq x_n \leq x_{M_N} + \frac{1}{N} \quad \forall\, n \geq M_N$$

$$y_n = \sup\{x_n, x_{n+1}, \ldots\}$$
$$z_n = \inf\{x_n, x_{n+1}, \ldots\}$$

$$x_{M_N} - \frac{1}{N} \leq z_{M_N} \leq y_{M_N} \leq x_{M_N} + \frac{1}{N}$$

$$\Rightarrow \quad y_{M_N} - z_{M_N} \leq \frac{2}{N}$$

$$\Rightarrow \quad \lim_{N\to\infty} (y_{M_N} - z_{M_N}) = 0$$

Thus, $\limsup\limits_{j\to\infty} x_j = \liminf\limits_{j\to\infty} x_j = x^*$

$$\Rightarrow \quad \lim_{j\to\infty} x_j = x^*$$

This property : "Any Cauchy sequence is convergent" is called completeness.

## Construction of the Reals

—OR—

## Completion of the Rational Numbers

In $\mathbb{Q}$, $\exists$ $\langle x_n \rangle \subseteq \mathbb{Q}$ that are Cauchy sequences which do not have rational limits.

For ~~To~~ each such Cauchy seq, we would add a new pt. to $\mathbb{Q}$

$\langle x_n \rangle$, $\langle x_n' \rangle$ are Cauchy seq in $\mathbb{Q}$

with $x_n - x_n' \longrightarrow 0$

$$\langle x_n \rangle - \langle x_n' \rangle \neq 0$$

Define an equivalence on Cauchy sequences

$$\langle x_n \rangle \sim \langle x_n' \rangle \text{ if } \lim_{n \to \infty} (x_n - x_n') = 0$$

For any equivalence relation, we must have:

(1) $\langle x_n \rangle \sim \langle x_n \rangle$ [Reflexivity]

(2) $\langle x_n \rangle \sim \langle x_n' \rangle$ then $\langle x_n' \rangle \sim \langle x_n \rangle$ [Symmetry]

(3) If $\langle x_n \rangle \sim \langle x_n' \rangle$ & $\langle x_n' \rangle \sim \langle x_n'' \rangle$ then $\langle x_n \rangle \sim \langle x_n'' \rangle$ [Transitivity]

## Why are these conditions important?

We define equivalence classes

$$[\langle x_n \rangle] = \{ \langle x_n' \rangle \mid \langle x_n' \rangle \sim \langle x_n \rangle \}$$

Because $\sim$ is an equivalence relation

$$[\langle x_n \rangle] = [\langle y_n \rangle]$$

$$\text{or} \quad [\langle x_n \rangle] \cap [\langle y_n \rangle] = \phi$$

The set of equivalence classes of Cauchy seqs. of rationals is defined to be the set underlying the real numbers.