09/12/17

Math 508 - Lecture 5

(Order 6) If $A \subseteq \mathbb{R}$ has an upper bound [A number c is an upper bd. for A if $\forall x \in A, x \leq c$], then A has a least upper bound.

Note: This axiom is not satisfied by $\mathbb{Q}$. (set of rationals)

- The additive inverse is unique.
- The multiplicative inverse is unique.
- The neutral elements 0, 1 are unique.
- $0 < a$ iff $(-a) < 0$

Pf: ($\Rightarrow$) $0 < a$

$$0 + (-a) < a + (-a)$$

$$(-a) < 0$$

[If $a < b$ then $a + c < b + c$ $\forall c \in \mathbb{R}$]

- $0 < 1$; $1 \neq 0$

Pf: By trichotomy of order

Either $\underbrace{0 < 1}_{\text{Done}}$ or $\underbrace{0 < -1}$

$$0 \cdot (-1) < (-1)(-1)$$

$0 < 1$, a contradiction.

Axiom: If $a < b$, $c > 0$ then $a \cdot c < b \cdot c$

- If $a \in \mathbb{R} \setminus \{0\}$
  then $a^2 > 0$

- If $0 < a < b$
  then $0 < \frac{1}{b} < \frac{1}{a}$

Q. Do the natural numbers $\mathbb{N}$ and the rational numbers $\mathbb{Q}$ exist in the real number system?

<u>Mapping</u>

$$\sigma : \mathbb{N} \longrightarrow \mathbb{R}$$

$$\mathbb{N} = \{1, 2, 3, 4\}$$

$$\sigma(1) = 1$$

$$\sigma(n) = 1 + \sigma(n-1)$$

← maps natural numbers into the real numbers

<u>Extend σ</u>

$$\sigma : \mathbb{Z} \longrightarrow \mathbb{R}$$

$$\sigma(0) = 0$$

$$\sigma(-n) = -\sigma(n)$$

← maps integers into $\mathbb{R}$

$$\mathbb{Z} \times \mathbb{R} \longrightarrow \mathbb{R}$$

$$\mathbb{1} \cdot a = a$$

$$\vdots$$

$$\mathbb{n} \cdot a = n \cdot a$$

$$-\mathbb{n} \cdot a = -n \cdot a$$

$\leftarrow$ all operations also ~~transform~~ in $\mathbb{R}$.

To the rational # $r = \frac{m}{n}$ $(n > 0)$

we assign the real #

$$\frac{m}{n} = \sigma(m) \cdot \sigma(n)^{-1} = r$$

(*) Map is from $\mathbb{Q} \to \mathbb{R}$, and not $\mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$

<u>Claim</u>: $\sigma(m \cdot p) \cdot \sigma(n \cdot p)^{-1} = \sigma(m) \cdot \sigma(n)$

<u>Pf</u>: Done in class

Call this map $T : \mathbb{Q} \longrightarrow \mathbb{R}$

Need to show that the mapping is from a field to a field, so we have a copy of $\mathbb{Q}$ in $\mathbb{R}$.

~~T(m/n)~~

Archimedean Axiom

If $0 < a < b$, then $\exists$ an $n \in \mathbb{N}$ such that $n \cdot a > b$

Proposition:

The least upper bound property implies the Archimedean axiom.

Proof:

Suppose not.

This means that $n \cdot a \leq b \quad \forall n \in \mathbb{N}$

-OR- equivalently $\{n \cdot a : n \in \mathbb{N}\}$ is bounded above by b.

The least upper bound property implies $\{n \cdot a : n \in \mathbb{N}\}$ has a l.u.b. c, a number s.t.

$$n \cdot a \leq c \quad \forall n \in \mathbb{N}$$

In this case $\exists\ n_0 \in \mathbb{N}$ s.t.

$$n_0 \cdot a > c - \frac{a}{2}$$

$$n_0 \cdot a + a > c + \frac{a}{2} \qquad \text{[add } a \text{ on both sides]}$$

$$(n_0+1)a > c + \frac{a}{2} > c \underset{\substack{\uparrow \\ c \text{ is an upper bound}}}{\geq} (n_0+2)a$$

$\Rightarrow \quad 0 > a$, a contradiction

■

Let $b = 1$

If $0 < a < 1$, then $\exists\, n \in \mathbb{N}$ such that $n.a > 1 \leftrightarrow a > \frac{1}{n}$

- If $a < \frac{1}{n}$ $\forall n \in \mathbb{N}$, then $a = 0$.

<u>Proposition</u>: $x \in \mathbb{R}$, $Q_x = \{ r \in \mathbb{Q} : r < x \}$

then l.u.b. $Q_x = x$

<u>Pf</u>: Suppose not.

Then $\exists$ $y < x$, s.t. $r \leq y$ $\forall y \in Q_x$

$0 < x - y$, $\exists$ an $n \in \mathbb{N}$ s.t. $\frac{1}{n} < x - y$

.... (complete as exercise)

<u>Proposition</u>: Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then there exists an $r \in \mathbb{Q}$ such that $0 < x - r < \frac{1}{n}$

<u>Proof</u>: Left to the reader.

<u>Consequence</u>: $\mathbb{Q}$ is dense in $\mathbb{R}$.

— OR — $\forall\, x \in \mathbb{R}$, $\exists\, r \in \mathbb{Q}$ such that ~~$n \in \mathbb{N}$~~
$n \in \mathbb{N}$

$$\boxed{r \leq x \leq r + \frac{1}{n}}$$

—OR—

Given $\varepsilon > 0$ $\exists$ a rational # r such that

$$r \leq x \leq r + \varepsilon$$

because $\exists\ n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

- The absolute value function is defined on $\mathbb{R}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{otherwise} \end{cases}$$

(1) $|x \cdot y| = |x| \cdot |y|$

(2) $|x+y| \leq |x| + |y|$

(3) $|x| \geq 0$; $|x| = 0$ iff $x = 0$

$\rightarrow$ can be used to define a distance on $\mathbb{R}$

$$d(x,y) = |x-y|$$

(1) $d(x,x) = 0$, $d(x,y) > 0$ iff $x \neq y$

(2) $d(x,y) = d(y,x)$

(3) $d(x,y) \leq d(x,z) + d(z,y)$

Given $x \in \mathbb{R}$, $n \in \mathbb{N}$, $\exists\ r \in \mathbb{Q}$ s.t.

$$\boxed{d(x,r) \leq \frac{1}{n}}$$

$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$

$\cancel{(n, x) \longrightarrow}$

- $(n, x) \longrightarrow n \cdot x$

defined recursively,

$$\left.\begin{aligned} &1 \cdot x \Rightarrow 1 \cdot x = x \\ &(n \cdot x) \longrightarrow (n-1) \cdot x + x = n \cdot x \end{aligned}\right\} \text{ multiplication}$$

- $(n, x) \longrightarrow x^n$

If $n > 0$

$(1, x) \longrightarrow x^1 = x$

$(n, x) \longrightarrow x^{n-1} \cdot x = x^n$

exponentiation

If $n < 0$

$(n, x) \longrightarrow [x^{|n|}]^{-1} = x^{-n}$

If $n = 0$

$\cancel{x = 0}$

$x^0 = 1$.

- $0 < x, y$

$x > y$ iff $x^n > y^n$

Defn:
If $a > 0$ and $n \in \mathbb{N}$, then x is an n^{th} root of a if $x^n = a$.

Do n^{th} roots exist?

Theorem: $\forall\ 0 < a,\ n \in \mathbb{N}$, $\exists$ a unique $x > 0$ such that $x^n = a$.

Proof:

(Uniqueness)

We will use:

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + y^{n-2}x + y^{n-1})$$

END OF LECTURE