The material covered in this problem set comes from Chapter 1 of Strichartz.

**Standard Problems**

These problems do not need to be handed in.

1. Show that there is no real number $x$ such that $x^2 = -1$.

2. Prove by induction: If $a_1, \ldots, a_n$ are real numbers, then
   \[
   |a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|. \tag{1}
   \]

3. Recall that for a pair of integers $(m, n)$ with $n \neq 0$, we define the equivalence class
   \[
   [(m, n)] = \{(m', n') : mn' = m'n\}. \tag{2}
   \]
   Show that for two pairs $(m, n), (p, q)$ either $[(m, n)] = [(p, q)]$ or $[(m, n)] \cap [(p, q)] = \emptyset$.

**Problems to Hand In**

Your solutions to the following problems should be carefully written up in English: Use complete sentences and paragraphs.

1. Let $P$ and $Q$ be statements; explain why the truth of $P \Rightarrow Q$ is equivalent to the truth of $\neg Q \Rightarrow \neg P$.

2. The set theoretic difference $A \setminus B$ is defined by
   \[
   A \setminus B = \{x \in A : x \notin B\}. \tag{3}
   \]
   If $A, B$ are subsets of the same “universe,” then show that $A \setminus B = A \cap B^c$. Prove the following formulæ for sets:
   \[
   (A \cap B)^c = A^c \cup B^c,
   (A \cup B)^c = A^c \cap B^c,
   A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C),
   A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C). \tag{4}
   \]
3. Remember that a set $A$ is countably infinite if there is a bijective map $f : \mathbb{N} \to A$.

Use mathematical induction to prove that every finite subset of $\mathbb{N}$ has a smallest element. (You may not use the well ordering principle!) Show that every subset of $\mathbb{N}$ is either finite, or countably infinite, and then show that there is no infinite set with cardinality less than that of $\mathbb{N}$.

4. Is the set of all finite subsets of $\mathbb{N}$ countable, or uncountable? You must prove your answer.

5. Suppose that $r = \frac{a}{b} < 1$ is a positive rational number. Let

$$a_1 = \min\{q \in \mathbb{N} : \frac{1}{q} \leq \frac{a}{b}\}.$$  (5)

(a) Show that $0 \leq aa_1 - b < a$.

(b) Recursively, for $1 \leq j$, assume that we have found $\{a_1, \ldots, a_j\}$, ($a_1$ is define above) and that

$$r - \frac{1}{a_1} - \cdots - \frac{1}{a_j} > 0,$$

then define

$$a_{j+1} = \min\{q \in \mathbb{N} : \frac{1}{q} \leq \frac{a}{b} - \frac{1}{a_1} - \cdots - \frac{1}{a_j}\}.$$  (6)

Show that $a_{j+1} > a_j$.

(c) Conclude that every rational number between 0 and 1 can be represented as

$$r = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$$

where $< a_j >$ is strictly increasing.

(d) Can you show that this representation is never unique?

6. The notation $\mathbb{Q}[x]$ denotes the set of polynomials in $x$ with rational coefficients.

(a) Prove that the set $\mathbb{Q}[x]$ is countable.

(b) A number $x$ that satisfies $p(x) = 0$ for a $p \in \mathbb{Q}[x]$ is called algebraic. How large is the the set of algebraic numbers?