Math 508
Problem set 11, due December 6, 2018
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 6.3, 6.4, 7.2 and 7.3 in **The Way of Analysis**.

You should do the following problems, but you do not need to hand in your solutions:

1. Prove that
$$\log 2 = \lim_{n\to\infty}\left(\frac{1}{n+1}+\cdots+\frac{1}{2n}\right).$$
2. If the improper Riemann integral
$$I = \int_{-\infty}^{\infty} f(x)dx$$
exists, show that, for any $y \in \mathbb{R}$ the improper integral
$$\int_{-\infty}^{\infty} f(x+y)dx$$
also exists and equals I.

The following problems should be carefully written up and handed in.

1. Suppose that f is positive and continuous on $(0, 1]$, and the improper Riemann integral
$$\int_0^1 f(x)dx = \lim_{\epsilon\to 0^+}\int_\epsilon^1 f(x)dx \tag{1}$$
exists. Show that this integral is the limit of the lower Riemann sums of f on $[0, 1]$, i.e., for any sequence of partitions $< P_i >$ of $[0, 1]$ with $|P_i| \to 0$,
$$\int_0^1 f(x)dx = \lim_{i\to\infty} S^-(f, P_i). \tag{2}$$
2. The function $\log x$ is defined by the integral:
$$\log x = \int_1^x \frac{dy}{y}.$$
   (a) Show that for any $0 < a$ we have that
$$\lim_{x\to\infty}\frac{\log x}{x^a} = 0 \text{ and } \lim_{x\to 0^+} x^a \log x = 0$$
   (b) Prove that $\log(1 + x) = x + O(x^2)$, for x near to zero. Hint:
$$\log(1+x) = \int_0^x \frac{dy}{1+y}. \tag{3}$$

3. For which values of a, b do the improper Riemann integrals exist:

$$\int_0^{\frac{1}{2}} |x|^a |\log x|^b dx?$$

4. Let f be a function in $\mathcal{C}^1([-1,1])$. Show that the following limit exists:

$$\lim_{\epsilon \to 0^+} \left[\int_\epsilon^1 \frac{f(x)dx}{x} + \int_{-1}^{-\epsilon} \frac{f(x)dx}{x} \right]. \tag{4}$$

5. (a) Suppose that $f_n(x) = x^n(1-x)$. Prove that $f_n(x) \to 0$ uniformly on $[0,1]$.
   (b) If $g_n(x) = n^2 x^n (1-x)$, then show that $g_n(x)$ converges pointwise to 0 for $x \in [0,1]$, but *not* uniformly.
6. Suppose that $< f_n >$ is a sequence of functions that converge uniformly to f. Suppose, moreover, that the $\lim_{x \to x_0} f_n(x)$ exists for every n. Prove that $\lim_{x \to x_0} f(x)$ exists and

$$\lim_{x \to x_0} f(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x).$$

Note: We are *not* assuming that the functions $\{f_n\}$ or f are continuous.