Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 6.3, 6.4, 7.2 and 7.3 in *The Way of Analysis*.

You should do the following problems, but you do not need to hand in your solutions:

1. Prove that
   \[ \log 2 = \lim_{n \to \infty} \left( \frac{1}{n+1} + \cdots + \frac{1}{2n} \right) . \]

2. If the improper Riemann integral
   \[ I = \int_{-\infty}^{\infty} f(x) \, dx \]
   exists, show that, for any \( y \in \mathbb{R} \) the improper integral
   \[ \int_{-\infty}^{\infty} f(x + y) \, dx \]
   also exists and equals \( I \).

The following problems should be carefully written up and handed in.

1. Suppose that \( f \) is positive and continuous on \((0, 1]\), and the improper Riemann integral
   \[ \int_{0}^{1} f(x) \, dx = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} f(x) \, dx \]
   exists. Show that this integral is the limit of the lower Riemann sums of \( f \) on \([0, 1]\), i.e., for any sequence of partitions \( < P_i > \) of \([0, 1]\) with \( |P_i| \to 0 \),
   \[ \int_{0}^{1} f(x) \, dx = \lim_{i \to \infty} S^{-}(f, P_i) . \]

2. The function \( \log x \) is defined by the integral:
   \[ \log x = \int_{1}^{x} \frac{dy}{y} . \]
   (a) Show that for any \( 0 < a \) we have that
   \[ \lim_{x \to \infty} \frac{\log x}{x^a} = 0 \text{ and } \lim_{x \to 0^+} x^a \log x = 0 \]
   (b) Prove that \( \log(1 + x) = x + O(x^2) \), for \( x \) near to zero. Hint:
   \[ \log(1 + x) = \int_{0}^{x} \frac{dy}{1 + y} . \]
3. For which values of $a, b$ do the improper Riemann integrals exist:

$$\int_0^1 |x|^a \log x|^b\,dx?$$

4. Let $f$ be a function in $C^1([-1, 1])$. Show that the following limit exists:

$$(4) \lim_{\varepsilon \to 0^+} \left[ \int_{\varepsilon}^1 \frac{f(x)\,dx}{x} + \int_{-1}^{-\varepsilon} \frac{f(x)\,dx}{x} \right].$$

5. (a) Suppose that $f_n(x) = x^n(1 - x)$. Prove that $f_n(x) \to 0$ uniformly on $[0, 1]$.

(b) If $g_n(x) = n^2 x^n(1 - x)$, then show that $g_n(x)$ converges pointwise to 0 for $x \in [0, 1]$, but not uniformly.

6. Suppose that $< f_n >$ is a sequence of functions that converge uniformly to $f$. Suppose, moreover, that the $\lim_{x \to x_0} f_n(x)$ exists for every $n$. Prove that $\lim_{x \to x_0} f(x)$ exists and

$$\lim_{x \to x_0} f(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x).$$

Note: We are not assuming that the functions $\{f_n\}$ or $f$ are continuous.