Math 508
Problem set 2, due September 19, 2017
Dr. Epstein

The material covered in class so far, and for the coming week is covered, in much greater detail, in Chapters 1–4 and Section 5.5, 5.6, 8.1, and 8.3 of *Analysis I*, by Tao.

Your solutions to these problems should be written in English: Use complete sentences and paragraphs. Explain your reasoning and why one formula follows from the previous one.

1. Using the axioms for an ordered field, prove the following inequalities:
   
   (a) If $x \neq 0$, then $(x^2 + y^2)/x^2 \geq 1$.
   
   (b) $2xy \leq x^2 + y^2$.
   
   (c) If $x > 0$, and $0 < y < 1$, then $x < x/y$.

2. Suppose that $< x_n >$ is a sequence of integers that converges to a limit. What can you say about such a sequence and its limit.

3. A sequence of real numbers $< x_n >$ is called a Cauchy sequence if for any $N \in \mathbb{N}$, there is an $M \in \mathbb{N}$ such that

   \[
   |x_n - x_m| \leq \frac{1}{N} \text{ provided that } n, m \geq M. \tag{1}
   \]

   Suppose that $< x_n >$ is a bounded, increasing sequence. Prove that it is also a Cauchy sequence. More generally show that if $< x_n >$ is a convergent sequence, then it is also a Cauchy sequence.

4. Let $x < y$ be real numbers. Prove that the set

   \[
   \{ r \in \mathbb{Q} : x < r < y \} \tag{2}
   \]

   is infinite. Is this set countable or uncountable? Explain your answer.

5. Let $< x_n >$ be a sequence of positive real numbers that converges to $x$, i.e.,

   \[
   \lim_{n \to \infty} x_n = x.
   \]

   Prove that $0 \leq x$.

6. Let $< r_n >$ be a sequence of positive rational numbers that converges to a rational number $r$. Suppose that $r_n = \frac{p_n}{q_n}$ in lowest terms. Show that either there exists an $N$ so that $r_n = r$ for $N \leq n$, or the set of numbers $\{q_n\}$ is unbounded.