Math 508  
Problem set 4, due October 2, 2018  
Dr. Epstein  

Your solutions to these problems should be written in English: Use complete sentences  
and paragraphs.  
For this week, the material comes from sections 3.1 and section 7.2 of *The Way of Analysis*. Standard problems, whose solutions do not need to be handed in.  
1. Let $A$ and $B$ be sets of real numbers. Show that  
$$\sup(A \cup B) \geq \sup A \text{ and } \sup(A \cap B) \leq \sup A.$$  
2. Prove that every subsequence of a subsequence of a sequence is also a subsequence of the original sequence.  

The following problems should be carefully written up and handed in:  

1. Let $< x_n >$ be a sequence of real numbers. Prove that  
$$\limsup_{n \to \infty} (-x_n) = -\liminf_{n \to \infty} x_n.$$  
2. Compute the sup, inf, lim sup, lim inf of the following sequences  
   (a) $x_n = \frac{1}{n} + (-1)^n$.  
   (b) $x_n = 1 + \frac{(-1)^n}{n}$.  
   (c) $x_n = (-1)^n + \frac{1}{n} + 2 \sin\left(\frac{n\pi}{2}\right)$.  
3. Suppose that $< x_n >$ and $< y_n >$ are sequences with finite lim sups. Show that  
$$\limsup_{n \to \infty} (x_n + y_n) \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n.$$  
   Give an example to show that this inequality is sometimes strict.  
4. Find a sequence of numbers whose set of limits points equals the positive integers.  
5. Can there exist a sequence whose set of limit points equals the set:  
   $$\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}?$$  
   Why or why not?  
6. Suppose that $\sum_{j=1}^{\infty} a_j$ is convergent, but not absolutely convergent. Let  
   (1) $a_j^+ = \max\{a_j, 0\}$ and $a_j^- = \min\{a_j, 0\}$.  
   Show that  
   (2) $\sum_{j=1}^{\infty} a_j^+ = \infty$ and $\sum_{j=1}^{\infty} a_j^- = -\infty$.  

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