Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, the material comes from Chapters 5 and 6 of *Analysis, I*, and Chapter 2 and section 3.1 of *The Way of Analysis*. Standard problems, whose solutions do not need to be handed in.

1. Let $A$ and $B$ be sets of real numbers. Show that
   \[ \sup(A \cup B) \geq \sup A \text{ and } \sup(A \cap B) \leq \sup A. \]

2. Say that two sequences are *equivalent* if they differ only in finitely many terms. That is $< x_n > \sim < y_n >$ if there exists an $M$ so that $x_n = y_n$ if $n \geq M$. Show that this defines an equivalence relation, and that equivalent sequences have the same limit points.

The following problems should be carefully written up and handed in:

1. Let $< x_n >$ be a sequence of real numbers. Prove that
   \[ \limsup_{n \to \infty} (-x_n) = -\liminf_{n \to \infty} x_n. \]

2. Compute the sup, inf, lim sup, lim inf of the following sequences
   (a) $x_n = \frac{1}{n} + (-1)^n$.
   (b) $x_n = 1 + \frac{(-1)^n}{n}$.
   (c) $x_n = (-1)^n + \frac{1}{n} + 2 \sin\left(\frac{n\pi}{2}\right)$.

3. Suppose that $< x_n >$ and $< y_n >$ are sequences with finite lim sups. Show that
   \[ \limsup_{n \to \infty} (x_n + y_n) \leq \limsup_{n \to \infty} x_n + \limsup_{n \to \infty} y_n. \]

   Give an example to show that this inequality is sometimes strict.

4. The Bolzano-Weierstrass theorem states that if $< x_k >$ is a bounded sequence of real numbers, then it has a convergent subsequence. Suppose that $< x_k >$ is a bounded sequence. Suppose that there is a number $x^*$, such that any subsequence of $< x_k >$ has a further subsequence that converges to $x^*$. Prove that
   \[ \lim_{j \to \infty} x_j = x^*. \]

5. Find a sequence of numbers whose set of limits points equals the positive integers.

6. Can there exist a sequence whose set of limit points equals the set:
   \[ \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}? \]

   Why or why not?