Math 508
Problem set 5, due October 8, 2017
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, the material comes from Chapters 7 of Analysis, I, and section 7.2 of The Way of Analysis.

Standard problems, whose solutions do not need to be handed in.

1. Suppose that \((a_1, \ldots, a_n)\) is a finite sequence, and \((m_1, \ldots, m_n)\) is a rearrangement of \((1, \ldots, n)\). Prove that

\[
\sum_{j=1}^{n} a_j = \sum_{j=1}^{n} a_{m_j}.
\]

That is the sum of a finite collection of numbers can be done in any order.

2. Show that if \((a_1, \ldots, a_n)\) is a finite sequence, then

\[
\left| \sum_{j=1}^{n} a_j \right| \leq \sum_{j=1}^{n} |a_j|.
\]

The following problems should be carefully written up and handed in:

1. Let \(< a_n >\) be a sequence with \( \lim_{n \to \infty} a_n = A \). Show that if \( b_n = a_{n+1} - a_n \), then the infinite series

\[
S = \sum_{n=1}^{\infty} b_n
\]

is convergent. What is \( S \) equal to?

2. Let \( 0 < r < 1 \), and let \( k \in \mathbb{N} \). Show that the series

\[
\sum_{n=1}^{\infty} n^k r^n
\]

is absolutely convergent. Hint: You will want to use the result of problem 5 on Problem Set 3. As part of this problem you will need to explain how to define rational powers, that is: how should we define \( x^{\frac{p}{q}} \), where \( 0 < x \in \mathbb{R} \), and \( p, q \in \mathbb{Z} \). Make sure you show that the value of \( x^{\frac{p}{q}} \) depends on the rational number \( \frac{p}{q} \) and not on the choice of the pair \( (p, q) \) used to represent it.
3. Let 
\[ \sum_{j=1}^{\infty} a_j \]
be a conditionally convergent series, that is *not* absolutely convergent. Prove that there is a rearrangement of the indices, \( \{n_j\} \), so that

\[ \lim_{m \to \infty} \sum_{j=1}^{m} a_{n_j} = +\infty. \]

Recall that a rearrangement, \( j \mapsto n_j \), is a 1-1 and onto map from \( \mathbb{N} \) to \( \mathbb{N} \).