Math 508  
Problem set 6, due October 17, 2017  
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.
The material for this week is not discussed in Analysis, I, but can be found in Chapter 3 of The Way of Analysis.

Standard problems, whose solutions do not need to be handed in.

1. Let \((a, b)\) and \((c, d)\) be two intervals. What are the possibilities for the intersection, \((a, b) \cap (c, d)\), and the union, \((a, b) \cup (c, d)\)?
2. Give an example to show that the infinite union of closed sets need not be closed.

The following problems should be carefully written up and handed in:

1. Give an example of two convergent series \(\sum_{j=1}^{\infty} a_j, \sum_{j=1}^{\infty} b_j\), such that

\[ \sum_{j=1}^{\infty} a_j b_j \]

is divergent. Is this possible if one of the series is absolutely convergent?
2. Suppose that \(\sum_{j=1}^{\infty} a_j\), and \(\sum_{j=1}^{\infty} b_j\) are absolutely convergent series and let

\[ (p_1, p_2, \ldots) \]

be any enumeration of the complete list of products \(a_i b_j\) for \(i = 1, \ldots, j = 1, \ldots\). Show that

\[ \sum_{j=1}^{\infty} p_j = \left( \sum_{j=1}^{\infty} a_j \right) \cdot \left( \sum_{j=1}^{\infty} b_j \right). \]

That is, you need to show that \(\sum_{j=1}^{\infty} p_j\) converges and gives the value on the right, no matter how the products \(< a_i b_j >\) are ordered.
3. Suppose that \(\sum_{j=1}^{\infty} a_j\) is conditionally convergent. Use the partial sum formula to show that \(\sum_{j=1}^{\infty} \frac{1}{j} a_j\) is also convergent.
4. Let \(A\) be a subset of \(\mathbb{R}\) and let \(A'\) be its set of limit points. Show that \(A'\) is a closed set.
5. Let \(U\) be an open set, show that \(U \setminus \{x_1, \ldots, x_n\}\) is always open. Is this true if we remove a countable subset from \(U\)?