Math 508
Problem set 7, due October 31, 2017
Dr. Epstein

For this week, read Sections 4.1 and 4.2 in *The Way of Analysis* and Chapter 9 in *Analysis, I*.

You should do the following problems, but you do not need to hand in your solutions:

1. Which subsets of $\mathbb{R}$ are both closed and open?
2. Let $A$ be an open set. When is the closure of $A$ a compact set?
3. Find the limit points of the set $\{n + \frac{1}{m}| m, n \in \mathbb{N}\}$.
4. Let $A \subset \mathbb{R}$. Show that $x$ is a limit point of $A$ if and only if there is a sequence $<x_n> \subset A$, with $x_n \neq x_m$ for $n \neq m$, such that $
lim_{n \to \infty} x_n = x$.
5. Show that a finite union of compact sets is compact.

The following problems should be carefully written up and handed in:

1. Let $A$ be a subset of $\mathbb{R}$ and define $d_A(x) = \inf\{|x - y| : y \in A\}$.
   For $\epsilon > 0$ define the set $A_\epsilon = \{x : d_A(x) < \epsilon\}$.
   Prove that $A_\epsilon$ is open. What is $\cap_{\epsilon > 0} A_\epsilon$?
2. Let $A \subset \mathbb{R}$ be a compact set with infinitely many points. Show that $A$ has a limit point.
3. Suppose that $A$ and $B$ are compact subsets of $\mathbb{R}$ such that $A \cap B = \emptyset$. Show that there are open sets $U \supset A$ and $V \supset B$ such that $U \cap V = \emptyset$.
4. Using the $\epsilon - \delta$ definition, prove that the function defined on $[0, \infty)$ by $f(x) = \sqrt{x}$ is continuous.
5. Suppose that is a function $f$ defined on $[0, 1]$ that satisfies
   $$|f(x) - f(y)| \leq 5|x - y|^{\frac{1}{3}}.$$ 
   Prove that $f$ is continuous and give an explicit formula for $\delta$ as a function of $\epsilon$.
6. Suppose that $f$ is a uniformly continuous function defined on $[0, 1] \cap \mathbb{Q}$. Show that there is a unique continuous function $F$ defined on $[0, 1]$ so that $F(x) = f(x)$ for every $x \in [0, 1] \cap \mathbb{Q}$.

Is this still true if we just assume that $f$ is continuous, but not uniformly continuous. Why or why not?