Math 508
Problem set 9, due November 13, 2018
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Sections 5.3-5.5 in The Way of Analysis.

You should do the following problems, but you do not need to hand in your solutions:
1. Suppose that $f$ is a 1-1 continuous function defined on an interval $(a, b)$. Show that $f$ is either strictly increasing, or strictly decreasing.
2. Define the function $x_C$ by
   
   \[
   x_C = \begin{cases} 
   x & \text{if } 0 \leq x, \\
   0 & \text{if } 0 > x.
   \end{cases}
   \]
   
   If $k$ is an integer greater than 1 then show that $x_C^k$ is a differentiable function on $\mathbb{R}$. What about the $k = 1$ case?
3. Show that a polynomial of positive, even order has either a global maximum, or a global minimum, but never both.
4. If $f$ is $C^2((a, b))$, then for $x \in (a, b)$ we have that
   \[
   \lim_{h \to 0} \frac{f(x + h) - 2f(x) + f(x - h)}{h^2} = f''(x).
   \]

The following problems should be carefully written up and handed in. In these problems we let $T_n(f, x_0, x)$ denote the $n$th order Taylor polynomial of $f$ at $x_0$

\[
T_n(f, x_0, x) = \sum_{j=0}^{n} \frac{f^{[j]}(x_0)}{j!} (x - x_0)^j.
\]

1. Show that, for $a < b$, the function $(x - a)^2_+ (b - x)^2_+$ is a continuously differentiable function, which is non-zero exactly on the interval $(a, b)$. Now prove that if $A \subset \mathbb{R}$ is a closed set, then there is a differentiable function that vanishes exactly on $A$. Hint: What is $A^C$?
2. For $n \in \mathbb{Z} \setminus \{0\}$ show that the functions $f_n : (0, \infty) \to (0, \infty)$ defined by $f_n(x) = x^n$ are invertible, and that the inverse functions, $g_n(x)$, are differentiable. Using the inverse function theorem, give the formula for $g'_n(x)$.
3. Suppose that $f \in C^n((a, b))$, and $T_n(f, x_0, x)$ is its $n$th order Taylor polynomial at $x_0$. Show that the first derivative of $T_n(f, x_0, x)$ with respect to $x$ is the $(n - 1)$st order Taylor polynomial of $f'$ at $x_0$, that is:
   \[
   \frac{d}{dx} T_n(f, x_0, x) = T_{n-1}(f', x_0, x).
   \]
4. Let \( f \) and \( g \) belong to \( C^n((a, b)) \), and let \( T_n(f(x_0, x)), T_n(g(x_0, x)) \) be their \( n \)th order Taylor polynomials.
   (a) Prove that the \( n \)th order Taylor polynomial of \( f + g \) is \( T_n(f(x_0, x)) + T_n(g(x_0, x)) \).
   (b) Prove that the \( n \)th order Taylor polynomial of \( f \cdot g \) is obtained from the product of polynomials, \( T_n(f(x_0, x)) \cdot T_n(g(x_0, x)) \), by dropping all terms of degree greater than \( n \).
   (c) Prove that the \( n \)th order Taylor polynomial of \( 1/(1 + x) \) at 0 is
   \[
   T_n(f, 0, x) = \sum_{j=0}^{n} (-1)^j x^j.
   \]

5. Let \( f \in C^n((a, b)) \).
   (a) If \( f \) has \( n + 1 \) distinct zeros in \((a, b)\), then prove that \( f^{[n]} \) has a least one zero in this interval.
   (b) Show that if \( f^{[n]} \) never vanishes in \((a, b)\), then \( f \) has at most \( n \) zeros, counted with multiplicity, in this interval.
   (c) Prove that a polynomial of degree \( n \) has at most \( n \) zeros.

6. Suppose that \( f \in C^n((a, b)) \) and its \( n \)th order Taylor polynomial, \( T_n(f(x_0, x)) \) is the same function for every \( x_0 \in (a, b) \). What can you say about \( f(x) \)?