CHAPTER 1

Linear algebra

Linear algebra arises throughout mathematics and the sciences. The most basic problem in linear algebra is to solve a system of linear equations. The unknown can be a vector in a finite-dimensional space, or an unknown function, regarded as a vector in an infinite-dimensional space. One of our goals is to present unified techniques which apply in both settings. The reader should be to some extent already familiar with most of the material in this chapter. Hence we will occasionally use a term before defining it precisely.

1. Introduction to linear equations

We begin by reviewing methods for solving both matrix equations and linear differential equations. Consider the linear equation \( Lx = b \), where \( b \) is a given vector, and \( x \) is an unknown vector. To be precise, \( x \) lives in a vector space \( V \) and \( b \) lives in a vector space \( W \); in some but not all situations, \( V = W \). The mapping \( L : V \to W \) is linear. We give precise definitions of vector spaces and linear maps a bit later.

There are many books on linear algebra; we recommend [HK] for its abstract treatment and [St] for its applications and more concrete treatment. This chapter includes several topics not generally found in any linear algebra book. To get started we consider three situations, two of them just fun, where linear algebra makes an appearance. After that we will be a bit more formal.

When there are finitely many equations in finitely many unknowns, we solve using row operations, or Gaussian elimination. We illustrate by discussing a simple Kakuro puzzle. Kakuro is a puzzle where one is given a grid, blank squares, the row sums, the column sums, and blacked-out squares. One is supposed to fill in the blanks in the grid, using the digits 1 through 9. There are two added constraints; no repeats (within a fixed group) are allowed and the solution is known to be unique. In difficult puzzles, knowing that the solution is unique can sometimes help. We ignore these constraints in the following simple example.

Consider a three by three grid, as shown in Figure 1. From left to right, the column sums are given to be 10, 13, and 13. From top to bottom, the row sums are given to be 17, 7, and 12. Since the solution is known to use only the whole numbers from 1 through 9, one can easily solve it. In this example, there are two possible solutions. Starting from the top and going across, the possible solutions are \((9, 8, 1, 2, 4, 3, 9)\) and \((8, 9, 2, 1, 4, 3, 9)\).

![Figure 1. An easy Kakuro puzzle](image)

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Label the unknown quantities $x_1, x_2, \ldots, x_7$. The given row and column sums lead to the following system of equations of six equations in seven unknowns:

\[
\begin{align*}
    x_1 + x_2 &= 17 \\
    x_3 + x_4 + x_5 &= 7 \\
    x_6 + x_7 &= 12 \\
    x_1 + x_3 &= 10 \\
    x_2 + x_4 + x_6 &= 13 \\
    x_5 + x_7 &= 13 
\end{align*}
\]

To solve this system of equations, we first write it in matrix form

\[
\begin{pmatrix}
    1 & 1 & 0 & 0 & 0 & 0 & 0 & 17 \\
    0 & 0 & 1 & 1 & 1 & 0 & 0 & 7 \\
    0 & 0 & 0 & 0 & 1 & 1 & 1 & 12 \\
    1 & 0 & 1 & 0 & 0 & 0 & 0 & 10 \\
    0 & 1 & 0 & 1 & 0 & 1 & 0 & 13 \\
    0 & 0 & 0 & 0 & 1 & 0 & 1 & 13 
\end{pmatrix}.
\]

(1)

The last column is the right-hand side of the system of equations, the other columns are the coefficients of the equations, and each row corresponds to an equation. It is convenient to reorder the equations, obtaining:

\[
\begin{pmatrix}
    1 & 1 & 0 & 0 & 0 & 0 & 0 & 17 \\
    1 & 0 & 1 & 0 & 0 & 0 & 10 \\
    0 & 1 & 0 & 1 & 0 & 1 & 0 & 13 \\
    0 & 0 & 1 & 1 & 1 & 0 & 0 & 7 \\
    0 & 0 & 0 & 0 & 1 & 0 & 1 & 13 \\
    0 & 0 & 0 & 0 & 0 & 1 & 1 & 12 
\end{pmatrix}.
\]

Reordering the equations does not change the solutions. It is easy to see that we can achieve any order by several steps in which we simply interchange two rows. Such an interchange is an example of an *elementary row operation*. These operations enable us to simplify a system of linear equations expressed in matrix form. There are three types of elementary row operations. We can add a multiple of one row to another row, we can interchange two rows, and we can multiply a row by a non-zero constant.

This simple example does not require much effort. By row operations, we get the equivalent system

\[
\begin{pmatrix}
    1 & 1 & 0 & 0 & 0 & 0 & 0 & 17 \\
    0 & -1 & 1 & 0 & 0 & 0 & 0 & -7 \\
    0 & 0 & 1 & 1 & 0 & 1 & 0 & 6 \\
    0 & 0 & 1 & 1 & 1 & 0 & 0 & 7 \\
    0 & 0 & 0 & 0 & 1 & 0 & 1 & 13 \\
    0 & 0 & 0 & 0 & 0 & 1 & 1 & 12 
\end{pmatrix}.
\]
After a few more simple steps we obtain
\[
\begin{pmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 1 & 16 \\
0 & 1 & 0 & 1 & 0 & 0 & -1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & -1 & -6 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 13 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 12
\end{pmatrix}.
\]
(2)

The matrix equation in (2) has the same solutions as the matrix equation in (1), but it is expressed in a manner where one can see the solutions. We start with the last variable and work backwards. No row in (2) determines the last variable \(x_7\), and hence this variable is arbitrary. The bottom row of (2) gives \(x_6 + x_7 = 12\) and hence determines \(x_6\) in terms of \(x_7\). The second row from the bottom determines \(x_5\) in terms of \(x_7\). We proceed, working backwards, and note that \(x_4\) is also arbitrary. The third row then determines \(x_3\) in terms of both \(x_4\) and \(x_7\). Using the information from all the rows, we write the general solution to (1) in vector form as follows:
\[
x = \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7
\end{pmatrix} = \begin{pmatrix} 16 \\ 1 \\ -6 \\ 0 \\ 13 \\ 12 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_7 \begin{pmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}
\]
(3)

Notice that there are two arbitrary parameters in the solution (3): the variables \(x_4\) and \(x_7\) can be any real numbers. Setting them both equal to 0 yields one particular solution. Setting \(x_7 = 9\) and \(x_4 = 1\), for example, gives the solution \(x = (8, 9, 2, 1, 4, 3, 9)\), which solves the Kakuro puzzle. Setting \(x_7 = 9\) and \(x_4 = 2\) gives the other Kakuro solution.

When solving a system of linear equations, three things can happen. There can be no solutions, one solution, or infinitely many solutions. See Theorem 2.5 below. Before proving it, we will need to introduce some useful concepts. We conclude this section with an example.

Example 1.1. Consider the system of three equations in three unknowns given by
\[
\begin{align*}
x_1 + 5x_2 - 3x_3 &= 2 \\
2x_1 + 11x_2 - 4x_3 &= 12 \\
x_2 + cx_3 &= b.
\end{align*}
\]
Here \(c\) and \(b\) are constants, and the form of the answer depends on them. In matrix form the system is
\[
\begin{pmatrix}
1 & 5 & -3 & 2 \\
2 & 11 & -4 & 12 \\
0 & 1 & c & b
\end{pmatrix}.
\]
Two row operations yield the equivalent system
\[
\begin{pmatrix}
1 & 5 & -3 & 2 \\
0 & 1 & 2 & 8 \\
0 & 0 & c-2 & b-8
\end{pmatrix}.
\]
If \(c = 2\), the last row says \(0 = 0x_1 + 0x_2 + 0x_3 = b - 8\). If \(b \neq 8\), then we get a contradiction and the system has no solution. If \(c = 2\) and \(b = 8\), then we can choose \(x_3\) arbitrarily and satisfy all three equations. If \(c \neq 2\), then we can solve the system whether or not \(b = 8\), and the solution is unique.
EXERCISE 1.1. Put $c = 1$ in Example 1.1 and solve the resulting system.

EXERCISE 1.2. Solve the Kakuro puzzle in Figure 2. (Don’t use linear equations!)

\begin{figure}[h]
\centering
\begin{tabular}{cccccc}
23 & 18 & 8 & 16 & 13 \\
30 & 18 & 8 & 13 \\
7 & 9 & \\
7 & \\
\end{tabular}
\caption{A kakuro puzzle}
\end{figure}

EXERCISE 1.3. Consider the linear system
\begin{align*}
  x + y + z &= 11 \\
  2x - y + z &= 13 \\
  y + 3z &= 19.
\end{align*}
Convert it to matrix form, solve by row operations, and check your answer.

EXERCISE 1.4. Consider the (augmented) linear system, expressed in matrix form, of four equations in six unknowns.

\[
\begin{pmatrix}
1 & 2 & 3 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 5 \\
\end{pmatrix}
\]

Find the most general solution, and write it in the form
\[ x = x_0 + \sum c_j v_j. \]

2. Vectors and linear equations

We begin an informal discussion of linear equations in general circumstances. This discussion needs to be somewhat abstract in order to allow us to regard linear differential equations and linear systems (matrix equations) on the same footing.

**Definition 2.1** (informal). A real vector space is a collection of objects called vectors. These objects can be added together, or multiplied by real scalars. These operations satisfy the axioms from Definition 5.2.
Definition 2.2. Let $V$ and $W$ be real vector spaces, and suppose $L : V \to W$ is a function. Then $L$ is called a linear transformation (or a linear map or linear) if for all $u, v \in V$, and for all $c \in \mathbb{R}$, we have

$$L(u + v) = Lu + Lv \quad (L.1)$$

$$L(cv) = cL(v). \quad (L.2)$$

Note that the addition on the left-hand side of (L.1) takes place in $V$, but the addition on the right-hand side in (L.1) takes place in $W$. The analogous comment applies to the scalar multiplications in (L.2).

The reader is surely familiar with the vector space $\mathbb{R}^n$, consisting of $n$-tuples of real numbers. Both addition and scalar multiplication are formed component-wise. See (7.1) and (7.2) below. Each linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by matrix multiplication. The particular matrix depends on the basis chosen, as we discuss in Section 8. Let $A$ be a matrix of real numbers with $n$ columns and $m$ rows. If $x$ is an element of $\mathbb{R}^n$, regarded as a column vector, then the matrix product $Ax$ is an element of $\mathbb{R}^m$. The mapping $x \mapsto Ax$ is linear. The notation $I$ can mean either the identity matrix or the identity linear transformation; in both cases $I(x) = x$ for each $x \in \mathbb{R}^n$.

While $\mathbb{R}^n$ is fundamental, it will be insufficiently general for the mathematics in this book. We will also need both $\mathbb{C}^n$ and various infinite-dimensional spaces of functions.

In Example 1.1, $V$ is the vector space $\mathbb{R}^3$. The linear map $L : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$L(x, y, z) = (x + 5y - 3z, 2x + 11y - 4z, y + cz).$$

We are solving the linear equation $L(x, y, z) = (2, 12, b)$.

When $V$ is a space of functions, and $L : V \to V$ is linear, we often say $L$ is a linear operator on $V$. The simplest linear operator is the identity mapping, sending each $v$ to itself; it is written $I$. Other well-known examples include differentiation and integration. For example, when $V$ is the space of all polynomials in one variable $x$, the map $D : V \to V$ given by $D(p) = p'$ is linear. The map $J : V \to V$ defined by $(Jp)(x) = \int_0^x p(t)dt$ is also linear.

Example 2.1. We consider the vector space of polynomials of degree at most three, and we define $L$ by $L(p) = p - p''$. We show how to regard $L$ as a matrix, although doing so is a bit clumsy. Put

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3.$$ 

Then we have

$$(Lp)(x) = p(x) - p''(x) = (a_0 - 2a_2) + (a_1 - 6a_3)x + a_2x^2 + a_3x^3.$$ 

There is a linear operator on $\mathbb{R}^4$, which we still write as $L$, defined by

$$L(a_0, a_1, a_2, a_3) = (a_0 - 2a_2, a_1 - 6a_3, a_2, a_3).$$

In matrix form, we have

$$L = \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (*)$$

Example 2.1 illustrates several basic points. The formula $L(p) = p - p''$ is much simpler than the matrix representation in formula $(*)$. In this book we will often use simple but abstract notation. One benefit is that different applied situations become mathematically identical. In this example we can think of the space $V_3$ of polynomials of degree at most three as essentially the same as $\mathbb{R}^4$. Thus Example 2.1 suggests the notions of dimension of a vector space and isomorphism.
Definition 2.3. Let $L : V \to V$ be a linear transformation. Then $L$ is called \textbf{invertible} if there are linear transformations $A$ and $B$ such that $I = LB$ and $I = AL$. A square matrix $M$ is \textbf{invertible} if there are square matrices $A$ and $B$ such that $I = MB$ and $I = AM$.

Remark. A linear map $L$ or a matrix $M$ is invertible if such $A$ and $B$ exist. If they both exist, then $A = B$:

$$A = AL = (AL)B = (AL)B = I = B.$$  

Note that we have used associativity of composition in this reasoning. When both $A$ and $B$ exist, we therefore write (without ambiguity) the inverse mapping as $L^{-1}$.

Examples where only one of $A$ or $B$ exists arise only in infinite-dimensional settings. See Example 5.3, which is well-known to calculus students when expressed in different words. It clarifies why we write $+C$ when we do indefinite integrals.

Definition 2.4. Let $L : V \to W$ be a linear transformation. The \textbf{null space} of $L$, written $\mathcal{N}(L)$, is the set of $v$ in $V$ such that $Lv = 0$. The \textbf{range} of $L$, written $\mathcal{R}(L)$, is the set of $w \in W$ such that there is some $v \in V$ with $Lv = w$.

Remark. Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be linear. Then $\mathcal{N}(L) = \{0\}$ if and only if $L$ is invertible. The conclusion does not hold if the domain and target spaces have different dimensions.

Theorem 2.5. Let $V, W$ be real vector spaces, and suppose $L : V \to W$ is linear. For the linear system $Lx = b$, exactly one of three possibilities holds:

1. For each $b \in V$, the system has a unique solution. When $W = V$, this possibility occurs if and only if $L$ is invertible, and the unique solution is $x = L^{-1}b$. In general, this possibility occurs if and only if $\mathcal{N}(L) = \{0\}$ (uniqueness) and $\mathcal{R}(L) = W$ (existence).

2. $b \notin \mathcal{R}(L)$. Then there is no solution.

3. $b \in \mathcal{R}(L)$, and $\mathcal{N}(L) \neq \{0\}$. Then there are infinitely many solutions. Each solution satisfies

$$x = \text{x}_\text{part} + v,$$

where $x_{\text{part}}$ is a particular solution and $v$ is an arbitrary element of $\mathcal{N}(L)$.

Proof. Consider $b \in W$. If $b$ is not in the range of $L$, then (2) holds. Thus we assume $b \in W$ is an element of $\mathcal{R}(L)$. Then there is an $x_{\text{part}}$ with $L(x_{\text{part}}) = b$. Let $x$ be an arbitrary solution to $L(x) = b$. Then $b = L(x) = L(x_{\text{part}})$. By linearity, we obtain

$$0 = L(x) - L(x_{\text{part}}) = L(x - x_{\text{part}}).$$

Hence $x - x_{\text{part}} \in \mathcal{N}(L)$. If $\mathcal{N}(L) = \{0\}$, then $x = x_{\text{part}}$ and (1) holds. If $\mathcal{N}(L) \neq \{0\}$, then we claim that (3) holds. We have verified (*) ; it remains to show that there are infinitely many solutions. Since $\mathcal{N}(L)$ is not 0 alone, and it is closed under scalar multiplication, it is an infinite set, and hence the number of solutions to $L(x) = b$ is infinite as well.

The main point of Theorem 2.5 is that a system of linear equations with real coefficients (in finitely or infinitely many variables) can have no solutions, one solution, or infinitely many solutions. There are no other possibilities. Analogous results hold for systems with complex coefficients and even more generally. See Exercises 2.5 through 2.8 for easy examples illustrating Theorem 2.5.

This idea of particular solution plus an element of the null space should be familiar from differential equations. We illustrate by solving an inhomogeneous, constant-coefficient, linear, ordinary differential equation (ODE).
EXAMPLE 2.2. We solve the differential equation $y'' + 5y' + 6y = 6x - 6$. Here we regard $y$ as a vector in an infinite-dimensional vector space of functions. Let $A(y) = y'' + 5y' + 6y$ and $b = 6x - 6$. Then $A$ is linear and we wish to solve $Ay = b$.

We first find the null space of $A$ by considering the homogeneous equation $A(y) = 0$. The standard approach is to assume a solution of the form $e^{\lambda x}$. We obtain the equation $(\lambda^2 + 5\lambda + 6)e^{\lambda x} = 0$.

Since $e^{\lambda x} \neq 0$, we can divide by it and obtain $\lambda^2 + 5\lambda + 6 = 0$. This equation is known as the characteristic equation for the differential equation. Here its solutions are $\lambda = -2, -3$. Hence the null space of $A$ is spanned by the two functions $e^{-2x}$ and $e^{-3x}$. Thus there are constants $c_1$ and $c_2$ such that $Ay = 0$ if and only if $y(x) = c_1 e^{-2x} + c_2 e^{-3x}$.

Next we need to find a particular solution. To do so, we seek a particular solution of the form $y = cx + d$. Then $y' = c$ and $y'' = 0$. Plugging in the equation and doing simple algebra yields $c = 1$ and $d = -\frac{11}{6}$.

The general solution to the differential equation is therefore

$$y(x) = c_1 e^{-2x} + c_2 e^{-3x} + x - \frac{11}{6}.$$ 

In a real world problem, we would have additional information, such as the values of $y$ and $y'$ at some initial point. We would then use these values to find the constants $c_1$ and $c_2$.

REMARK. In engineering, the current is often written $i(t)$ or $I(t)$. For us $i$ will denote the imaginary unit ($i^2 = -1$) and $I$ will denote the identity operator or identity matrix. Hence we write $I$ for the current, both in the next example and in later chapters.

EXAMPLE 2.3. The current in an RLC circuit is governed by a second-order, constant-coefficient, linear ordinary differential equation.

![RLC circuit](image3)

**Figure 3.** An RLC circuit

Three circuit elements, or impedances, arise in the coefficients of this ODE. Let $E(t)$ be the voltage drop at time $t$, and let $I(t)$ be the current at time $t$. For a resistor, Ohm’s law yields

$$E_{\text{resistor}}(t) = R I(t).$$

The constant $R$ is called the **resistance**. Hence $E'_{\text{resistor}} = R I'$. 


For an inductor, the voltage drop is proportional to the time derivative of the current:

\[ E_{\text{inductor}}(t) = LI'(t). \]

The constant of proportionality \( L \) is called the \textbf{inductance}. Hence \( E'_{\text{inductor}} = LI'' \).

For a capacitor, the voltage drop is proportional to the charge \( Q(t) \) on the capacitor, and the current is the time derivative of the charge:

\[ E_{\text{capacitor}}(t) = \frac{1}{C}Q(t). \]

\[ E'_{\text{capacitor}}(t) = \frac{1}{C}I(t). \]

The constant \( C \) is called the \textbf{capacitance}.

Using Kirchhoff’s laws, which follow from conservation of charge and conservation of energy, and assuming linearity, we add the three contributions to obtain the time derivative of the voltage drop:

\[ LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t). \]

The differential equation (4) expresses the unknown current in terms of the voltage drop, the inductance, the resistance, and the capacitance. The equation is linear and of the form \( A(I) = b \). Here \( A \) is the operator given by

\[ A = L \frac{d^2}{dt^2} + R \frac{d}{dt} + \frac{1}{C}I. \]

The domain of \( A \) consists of a space of functions. The current \( t \mapsto I(t) \) lives in some infinite-dimensional vector space of functions. We find the current by solving the linear system \( A(I) = b \).

\textbf{Remark.} The characteristic equation of the differential equation for an RLC circuit is

\[ L\lambda^2 + R\lambda + \frac{1}{C} = 0. \]

Since \( L, R, C \geq 0 \), the roots have non-positive real part. Consider the three cases

\[ R^2 - \frac{4L}{C} > 0 \]

\[ R^2 - \frac{4L}{C} = 0 \]

\[ R^2 - \frac{4L}{C} < 0. \]

These three cases correspond to whether the roots of the characteristic equation are real and distinct, repeated, or complex. The expression \( \zeta = \sqrt{\frac{C}{L} R} \) is called the \textit{damping factor} of this circuit. Note that \( \zeta > 1 \) if and only if \( R^2 - \frac{4L}{C} > 0 \) and \( \zeta < 1 \) if and only if \( R^2 - \frac{4L}{C} < 0 \).

The case \( \zeta > 1 \) is called \textit{over-damped}. The case \( \zeta = 1 \) is called \textit{critically damped}. The case \( \zeta < 1 \) is called \textit{under-damped} and we get oscillation.

See Figure 4. The discriminant \( R^2 - \frac{4L}{C} \) has units ohms squared; the damping factor is a dimensionless quantity.

\textbf{Exercise 2.1.} What is the null space of the operator \( D^3 - 3D^2 + 3D - I \)? Here \( D \) denotes differentiation, defined on the space of infinitely differentiable functions.

\textbf{Exercise 2.2.} Find the general solution to the ODE \( y'' - y = x \).

\textbf{Exercise 2.3.} Find the general solution to the ODE \( y' - ay = e^x \). What happens when \( a = 1 \)?
3. Matrices and row operations

We wish to extend our discussion of row operations. We begin by recalling the definitions of matrix addition and matrix multiplication. We add two matrices of the same size by adding the corresponding elements. The $i,j$ entry in $A + B$ is therefore given by

$$(A + B)_{ij} = A_{ij} + B_{ij}.$$  

We can multiply matrices $A$ and $B$ to form $BA$ only when the number of columns of $B$ equals the number of rows of $A$. In that case, the $i,j$ entry in $BA$ is defined by

$$(BA)_{ij} = \sum_{k=1}^{n} B_{ik} A_{kj}.$$ 

Here $A$ has $n$ columns and $B$ has $n$ rows. It follows easily from this definition that matrix multiplication is associative: $A(BC) = (AB)C$ whenever all the products are defined. Matrix multiplication is not generally commutative; even if both $AB$ and $BA$ are both defined, they need not be equal. We return to these issues a bit later.

Exercise 2.4. For what value of $C$ is the circuit in eqn. (4) critically damped if $L = 1$ and $R = 4$?

Exercise 2.5. Consider the system of two equations in two unknowns given by $x + 2y = 7$ and $2x + 4y = 14$. Find all solutions and express them in the form (3) from Section 1. For what values of $a, b$ does the system $x + 2y = a$ and $2x + 4y = b$ have a solution?

Exercise 2.6. Consider the system $x + 2y = 7$ and $2x + 6y = 16$. Find all solutions. For what values of $a, b$ does the system $x + 2y = a$ and $2x + 6y = b$ have a solution?

Exercise 2.7. In this exercise $i$ denotes $\sqrt{-1}$. Consider the system $z + (1+i)w = -1$ and $iz - w = 0$, for $z, w$ unknown complex numbers. Find all solutions.

Exercise 2.8. Again $i$ denotes $\sqrt{-1}$. Consider the system $z + (1+i)w = -1$ and $-iz + (1-i)w = i$. Find all solutions and express in the form (3) from Section 1.
A matrix with \( m \) columns and \( n \) rows (with real entries) defines a linear map \( A : \mathbb{R}^m \to \mathbb{R}^n \) by matrix multiplication: \( x \mapsto Ax \). If \( x \in \mathbb{R}^m \), then we can regard \( x \) as a matrix with 1 column and \( m \) rows. The matrix product \( Ax \) is defined and has 1 column and \( n \) rows. We identify such a column vector with an element of \( \mathbb{R}^n \).

When we are solving the system \( Ax = b \), we augment the matrix by adding an extra column corresponding to \( b \):

\[
\begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1m} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2m} & b_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nm} & b_n
\end{pmatrix}
\]

We perform row operations on a matrix, whether augmented or not. Let \( r_j \) denote the \( j \)-th row of a given matrix. The row operations are

- Multiply a row by a non-zero constant: \( r_j \mapsto \lambda r_j \) where \( \lambda \neq 0 \).
- Interchange two rows. For \( k \neq j \), \( r_j \mapsto r_k \) and \( r_k \mapsto r_j \).
- Add a multiple of a row to another row. For \( \lambda \neq 0 \) and \( k \neq j \), \( r_j \mapsto r_j + \lambda r_k \).

Doing a row operation on a matrix \( A \) corresponds to multiplying on the left by a particular matrix, called an \textit{elementary row matrix}. We illustrate each of the operations in the 2-dimensional case:

\[
\begin{pmatrix}
1 & 0 \\
0 & \lambda
\end{pmatrix}
\]  
\[\text{(ER.1)}\]

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]  
\[\text{(ER.2)}\]

\[
\begin{pmatrix}
1 & \lambda \\
0 & 1
\end{pmatrix}
\]  
\[\text{(ER.3)}\]

Multiplying the second row of a (two-by-two) matrix corresponds to multiplying on the left by the matrix in (ER.1). Switching two rows corresponds to multiplying on the left by the matrix in (ER.2). Adding \( \lambda \) times the second row to the first row corresponds to multiplying on the left by the matrix in (ER.3).

The language used to describe linear equations varies considerably. When mathematicians say that \( Lu = b \) is a \textit{linear equation}, \( L \) can be an arbitrary linear map between arbitrary vector spaces. We regard \( b \) as known, and we seek all solutions \( u \). Here \( u \) is the variable, or the unknown. It could be a column vector of \( n \) unknown numbers, or an unknown function, regarded as a vector in an infinite-dimensional space of functions. When \( u \) is a column vector of \( n \) unknown numbers and \( b \) consists of \( m \) known numbers, we can regard the linear equation \( Lu = b \) as \( m \) linear equations in \( n \) unknowns. The term \textit{system of linear equations} is often used in this case. The author prefers thinking of a system as a single equation for an unknown vector.

Row operations enable us to replace a linear equation with another linear equation whose solutions are the same, but which is easier to understand.

\textbf{Definition 3.1.} Let \( L : V \to W_1 \) and \( M : V \to W_2 \) be linear transformations. The linear equations \( Lu = b \) and \( Mv = c \) are \textit{equivalent} if they have the same solution sets.

In Definition 3.1, the linear maps \( L \) and \( M \) must have the same domain, but not necessarily the same target space. The reason is a bit subtle; the definition focuses on the solution set. Exercise 3.3 shows for example that a system of two linear equations in three unknowns can be equivalent to a system of three linear
equations in three unknowns. In such a situation one of the three equations is redundant. These ideas will be clarified when we formalize topics such as linear dependence and dimension.

Row operations on a matrix enable us to replace a given linear equation with an equivalent equation, which we hope is easier to understand. When \( L \) is invertible, the equation \( Lu = b \) is equivalent to the equation \( u = L^{-1}b \), which exhibits the unique solution. We repeat that a row operation amounts to multiplying on the left by an invertible matrix, and doing so does not change the set of solutions to the equation.

When solving a system, one performs enough row operations to be able to read off the solutions. One need not reach either of the forms in the following definition, but the terminology is nonetheless useful.

**Definition 3.2.** A matrix is **row-reduced** if the first entry (from the left) in each non-zero row equals 1, and all the other entries in the column corresponding to such an entry equal 0. For each non-zero row, the 1 in the first entry is called a **leading 1**. A matrix is in **row echelon form** if it is row-reduced and, in addition, the following hold:

- Each zero row is below all of the non-zero rows.
- The non-zero rows are ordered as follows: if the leading 1 in row \( j \) occurs in column \( c_j \), then \( c_1 < c_2 < \cdots < c_k \).

**Example 3.1.** The following matrices are in row echelon form:

\[
\begin{pmatrix}
0 & 1 & 2 & 0 & 5 \\
0 & 0 & 0 & 1 & 4 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The following matrix is not in row-echelon form:

\[
\begin{pmatrix}
0 & 1 & 2 & 0 & 5 \\
0 & 1 & 0 & 1 & 4
\end{pmatrix}
\]

The matrix (1) from the Kakuro problem is not row-reduced; the matrix (2) is row-reduced but not in row echelon form.

It is important to understand that \( AB \) need not equal \( BA \) for matrices \( A \) and \( B \). When \( AB = BA \) we say that \( A \) and \( B \) **commute**. The difference \( AB - BA \) is called the **commutator** of \( A \) and \( B \); it plays a significant role in quantum mechanics. See Chapter 7. An amusing way to regard the failure of commutivity is to think of \( B \) as putting on your socks and \( A \) as putting on your shoes. Then the order of operations matters. Furthermore, we also see (when \( A \) and \( B \) are invertible) that \((AB)^{-1} = B^{-1}A^{-1}\). Put on your socks, then your shoes. To undo, first take off your shoes, then your socks.

We have observed that matrix multiplication is associative. The identity

\[
(AB)C = A(BC)
\]

holds whenever the indicated matrix products are defined; it follows easily from the definition of matrix multiplication. If we regard matrices as linear maps, then associativity holds because composition of functions is always associative.

If some sequence of elementary row operations takes \( A \) to \( B \), then we say that \( A \) and \( B \) are **row-equivalent**. Since each elementary row operation has an inverse, it follows that row-equivalence is an equivalence relation. We discuss equivalence relations in detail in the next section.

**Exercise 3.1.** Prove that matrix multiplication is associative.
EXERCISE 3.2. Write the following matrix as a product of elementary row matrices and find its inverse:

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 7 \\
0 & 1 & 0
\end{pmatrix}
\]

EXERCISE 3.3. Determine whether the linear equations are equivalent:

\[
\begin{pmatrix}
1 & 2 & 3 \\
1 & 2 & 4 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
6 \\
7 \\
1
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= 
\begin{pmatrix}
6 \\
1
\end{pmatrix}
\]

EXERCISE 3.4. In each case, factor \( A \) into a product of elementary matrices. Check your answers.

\[
A = \begin{pmatrix}
6 & 1 \\
-3 & -1
\end{pmatrix}
\]

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

EXERCISE 3.5. Find 2-by-2 matrices \( A, B \) such that \( AB = 0 \) but \( BA \neq 0 \).

EXERCISE 3.6. When two impedances \( Z_1 \) and \( Z_2 \) are connected in series, the resulting impedance is \( Z = Z_1 + Z_2 \). When they are connected in parallel, the resulting impedance is

\[
Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}} = \frac{Z_1 Z_2}{Z_1 + Z_2}.
\]

A traveler averages \( v_1 \) miles per hour going one direction, and returns (exactly the same distance) averaging \( v_2 \) miles per hour. Find the average rate for the complete trip. Why is the answer analogous to the answer for parallel impedances?

4. Equivalence relations

In order to fully understand linear systems, we need a way to say that two linear systems, perhaps expressed very differently, provide the same information. Mathematicians formalize this idea using equivalence relations. This concept appears in both pure and applied mathematics; we discuss it in detail and provide a diverse collection of examples.

Sometimes we are given a set, and we wish to regard different members of the set as the same. Perhaps the most familiar example is parity. Often we care only whether a whole number is even or odd. More generally, consider the integers \( \mathbb{Z} \) and a modulus \( p \) larger than 1. Given integers \( m, n \) we regard them as the same if \( m - n \) is divisible by \( p \). For example, \( p \) could be 12 and we are thinking of clock arithmetic. To make this sort of situation precise, mathematicians introduce equivalence relations and equivalence classes.

First we need to define relation. Let \( A \) and \( B \) be arbitrary sets. Their Cartesian product \( A \times B \) is the set of ordered pairs \( (a, b) \) where \( a \in A \) and \( b \in B \). A relation \( R \) is an arbitrary subset of \( A \times B \). A function \( f : A \rightarrow B \) can be regarded as a special kind of relation; it is the subset consisting of pairs of the form \( (a, f(a)) \). Another example of a relation is inequality. Suppose \( A = B = \mathbb{R} \); we say that \( (a, b) \in R \)
if, for example, \( a < b \). This relation is not a function, because for each \( a \) there are many \( b \) related to \( a \). A function is a relation where, for each \( a \), there is a unique \( b \) related to \( a \).

We use the term relation on \( S \) for a subset of \( S \times S \). We often write \( x \sim y \) instead of \( (x, y) \in R \).

**Definition 4.1.** An equivalence relation on a set \( S \) is a relation \( \sim \) such that

- \( x \sim x \) for all \( x \in S \). (reflexive property)
- \( x \sim y \) implies \( y \sim x \). (symmetric property)
- \( x \sim y \) and \( y \sim z \) implies \( x \sim z \). (transitive property)

An equivalence relation partitions a set into equivalence classes. The equivalence class containing \( x \) is the collection of all objects equivalent to \( x \). Thus an equivalence class is a set.

Equality of numbers (or other objects) provides the simplest example of an equivalence relation. We give several additional examples next, emphasizing those from linear algebra.

**Example 4.1.** Congruence provides a good example of an equivalence relation. Let \( \mathbb{Z} \) be the set of integers, and let \( p \) be a modulus. We write \( m \sim n \) if \( m - n \) is divisible by \( p \). For example, if \( p = 2 \), then there are two equivalence classes, the odd numbers and the even numbers. For example, if \( p = 3 \), then there three equivalence classes: the set of whole numbers divisible by three, the set of whole numbers one more than a multiple of three, and the set of whole numbers one less than a multiple of three.

**Example 4.2.** Let \( A \) and \( B \) be matrices of the same size. We say \( A \) and \( B \) are **row-equivalent** if there is a sequence of row operations taking \( A \) to \( B \).

**Example 4.3.** Square matrices \( A \) and \( B \) are called **similar** if there is an invertible matrix \( P \) such that \( A = PBP^{-1} \). Similarity is an equivalence relation.

**Example 4.4.** In Chapter 6 we say that two functions are equivalent if they agree except on a small set (a set of measure zero). An element of a Hilbert space will be equivalence class of functions, rather than a function itself.

In this book we presume the real numbers are known. One can however regard the natural numbers \( \mathbb{N} \) as the starting point. Then one constructs the integers \( \mathbb{Z} \) from \( \mathbb{N} \), the rational numbers \( \mathbb{Q} \) from \( \mathbb{Z} \), and finally the real number system \( \mathbb{R} \) from \( \mathbb{Q} \). Each of these constructions involves equivalence classes. We illustrate for the rational number system.

**Example 4.5 (Fractions).** To construct \( \mathbb{Q} \) from \( \mathbb{Z} \), we consider ordered pairs of integers \((a, b)\) with \( b \neq 0 \). The equivalence relation here is that \((a, b) \sim (c, d)\) if \( ad = bc \). Then \((1, 2) \sim (−1, −2) \sim (2, 4)\) and so on. Let \( \frac{a}{b} \) denote the equivalence class of all pairs equivalent to \((a, b)\). Thus a rational number is an equivalence class of pairs of integers! We define addition and multiplication of these equivalence classes. For example, \( \frac{2}{3} + \frac{1}{n} \) is the class containing \((an + bn, bn)\). See Exercise 4.5.

An equivalence class is a set, but thinking that way is sometimes too abstract. We therefore often wish to choose a particular element, or representative, of a given equivalence class. For example, if we think of a rational number as an equivalence class (and hence infinitely many pairs of integers), then we might select as a representative the fraction that is in lowest terms.

In linear algebra, row equivalence provides a compelling example of an equivalence relation. Exercise 4.1 asks for the easy proof. Since an invertible matrix is row equivalent to the identity matrix, we could represent the equivalence class of all invertible matrices by the identity matrix. Later in this chapter we show that a matrix is diagonalizable if and only if it is similar to a diagonal matrix. In this case the diagonal matrix is often the most useful representative of the equivalence class.

Most linear algebra books state the following result, although the language used may differ. The row-equivalence class containing the identity matrix consists of the invertible matrices.
Theorem 4.2. Let $A$ be a square matrix of real (or complex numbers). Then the following statements all hold or all fail simultaneously.

- $A$ is invertible.
- $A$ is row equivalent to the identity matrix.
- $A$ is a product of elementary matrices.

We illustrate this theorem by factoring and finding the inverse of the following matrix:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$  

We perform row operations on $A$ until we reach the identity:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

At each step we multiply on the left by the corresponding elementary matrix and express the process by a product of matrices:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

We think of (5) as $E_4E_3E_2E_1A = I$, where the $E_j$ are elementary row matrices. Thus $A^{-1} = E_4E_3E_2E_1$. Hence

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$  

Taking the inverse matrices and writing the product in the reverse order yields $A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$ as the factorization of $A$ into elementary matrices:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

When finding inverses alone, one can write the same steps in a more efficient fashion. Augment $A$ by including the identity matrix to the right, and apply the row operations to this augmented matrix. When the left half becomes the identity matrix, the right half is $A^{-1}$. In this notation we get

$$\begin{pmatrix} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{pmatrix}.$$  

Thus row operations provide a method both for finding $A^{-1}$ and for factoring $A$ into elementary matrices. This reasoning provides the proof of Theorem 4.2, which we leave to the reader.

We conclude this section with an amusing game. The solution presented here illustrates the power of geometric thinking; orthogonality and linear independence provide the key idea.

**Example 4.6.** There are two players $A$ and $B$. Player $A$ names the date January 1. Player $B$ must then name a date, later in the year, changing only one of the month or the date. Then player $A$ continues in the same way. The winner is the player who names December 31. If player $B$ says January 20, then player $B$ can force a win. If player $B$ says any other date, then player $A$ can force a win.

We work in two dimensional space, writing $(x, y)$ for the month and the day. The winning date is $(12, 31)$. Consider the line given by $y = x + 19$. Suppose a player names a point $(x, y)$ on this line. According to the rules, the other player can name either $(x + a, y)$ or $(x, y + b)$, for $a$ and $b$ positive. Neither
of these points lies on the line. The player who named \((x, y)\) can then always return to the line. Thus a player who names a point on this line can always win. Hence the only way for player B to guarantee a win on the first turn is to name January 20, the point \((1, 20)\).

![Figure 5. Solution of the game](image)

**Exercise 4.1.** Prove that row equivalence is an equivalence relation.

**Exercise 4.2.** Prove Theorem 4.2.

**Exercise 4.3.** Describe all row equivalence classes of 2-by-2 matrices.

**Exercise 4.4.** Verify that the three properties of an equivalence relation hold in Example 4.5. Why is this definition used? What is the definition of multiplication of rational numbers?

**Exercise 4.5.** Let \(S\) be the set of students at a college. For \(a, b \in S\) write \(a \sim b\) if \(a\) and \(b\) have taken a class together. Is this relation an equivalence relation?

**Exercise 4.6.** (Dollars and sense). For real numbers define \((x, y) \sim (a, b)\) if \(x + 100y = a + 100b\). Is this relation an equivalence relation? Give a simple interpretation!

**Exercise 4.7.** Determine all 2-by-2 matrices that are row equivalent to \(
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\).

**Exercise 4.8.** There is a pile of 1000 pennies for a two-person game. The two players alternate turns. On your turn you may remove 1, 2, or 3 pennies. The person who removes the last penny wins the game. Which player wins? What if the original pile had 1001 pennies?

**Exercise 4.9.** Which of the following subsets of \(\mathbb{R}^2\), thought of as relations on \(\mathbb{R}\), define functions?

- The set of \((x, y)\) with \(x + y = 1\).
- The set of \((x, y)\) with \(x^2 + y^2 = 1\).
- The set of \((x, y)\) with \(x^3 + y^3 = 1\).
5. Vector spaces

We recall that \( \mathbb{R}^n \) consists of \( n \)-tuples of real numbers. Thus we put
\[
\mathbb{R}^n = \{(x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R}\}
\]
\[
\mathbb{C}^n = \{(z_1, z_2, \cdots, z_n) : z_i \in \mathbb{C}\}
\]
In both cases, we define addition and scalar multiplication componentwise:
\[
(u_1, u_2, \cdots, u_n) + (v_1, v_2, \cdots, v_n) = (u_1 + v_1, u_2 + v_2, \cdots, u_n + v_n) \quad (7.1)
\]
\[
c(u_1, u_2, \cdots, u_n) = (cu_1, cu_2, \cdots, cu_n) \quad (7.2)
\]
In both cases, the axioms for a vector space hold. Thus \( \mathbb{R}^n \) and \( \mathbb{C}^n \) will be two of our most important examples of vector spaces. It is standard notation within mathematics to regard the \( n \)-tuple \((x_1, \ldots, x_n)\) as the same object as the column vector
\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
\]
The (column) vector \((x_1, x_2, \ldots, x_n)\) and the row vector \((x_1 \ x_2 \ \ldots \ x_n)\) are not considered the same object.
We return to this point in Chapter 3 when we introduce dual spaces.

For \( \mathbb{R}^n \), the scalars are real numbers. For \( \mathbb{C}^n \), the scalars will be complex numbers. The collection of scalars forms what is known in mathematics as the ground field or scalar field.

We give the formal definition of vector space. First we need to define field. The word field has two distinct uses in mathematics. Its use in terms such as vector field is completely different from its use in the following definition. In nearly all our examples, the ground field will be \( \mathbb{R} \) or \( \mathbb{C} \), although other fields also arise in various applications.

Definition 5.1. A field \( F \) is a mathematical system consisting of a set of objects (often called scalars) together with two operations, addition and multiplication, satisfying the following axioms. As usual we write \( xy \) for multiplication, instead of a notation such as \( x \cdot y \).

- There are distinct elements of \( F \) written 0 and 1.
- For all \( x, y \) we have \( x + y = y + x \).
- For all \( x, y, z \) we have \( (x + y) + z = x + (y + z) \).
- For all \( x \) we have \( x + 0 = x \).
- For all \( x \) there is a \(-x\) such that \( x + (-x) = 0 \).
- For all \( x, y \) we have \( xy = yx \).
- For all \( x, y, z \) we have \( (xy)z = x(yz) \).
- For all \( x \) we have \( x1 = x \).
- For all \( x \neq 0 \) there is an \( x^{-1} \) such that \( xx^{-1} = 1 \).
- For all \( x, y, z \) we have \( x(y + z) = xy + xz \).

Although these axioms might seem tedious, they are easy to remember. A field is a mathematical system in which one can add, subtract, multiply, and divide (except that one cannot divide by zero), and the usual laws of arithmetic hold. The basic examples for us are the real numbers \( \mathbb{R} \) and the complex numbers \( \mathbb{C} \). The rational numbers \( \mathbb{Q} \) also form a field, as does the set \( \{0, 1, \ldots, p - 1\} \) under modular arithmetic, when \( p \) is a prime number. When \( p \) is not prime, factors of \( p \) (other than 1) do not have a multiplicative inverse.
**Remark.** Two elementary facts about fields often arise. The first fact states that \( x0 = 0 \) for all \( x \). This result follows by writing
\[
x0 = x(0 + 0) = x0 + x0.
\]
Adding the additive inverse \(- (x0)\) to both sides yields 0 = \( x0 \). The second fact is that \( xy = 0 \) implies that at least one of \( x \) and \( y \) is itself 0. The proof is simple. If \( x = 0 \), the conclusion holds. If not, multiplying both sides of \( 0 = xy \) by \( x^{-1} \) and then using the first fact yield \( 0 = y \).

**Remark.** Both the rational numbers and the real numbers form ordered fields. In an ordered field, it makes sense to say \( x > y \) and one can work with inequalities as usual. The complex numbers are a field but not an ordered field. It does not make sense to write \( z > w \) if \( z \) and \( w \) are non-real complex numbers.

**Definition 5.2.** Let \( F \) be a field. A vector space \( V \) over \( F \) is a set of vectors together with two operations, addition and scalar multiplication, such that

1. There is an object \( 0 \) such that, for all \( v \in V \), we have \( v + 0 = v \).
2. For all \( u, v \in V \) we have \( u + v = v + u \).
3. For all \( u, v, w \in V \) we have \( u + (v + w) = (u + v) + w \).
4. For all \( v \in V \) there is a \(- v \) such that \( v + (- v) = 0 \).
5. For all \( v \in V \), we have \( 1v = v \).
6. For all \( c \in F \) and all \( v, w \in V \) we have \( c(v + w) = cv + cw \).
7. For all \( c_1, c_2 \in F \) and all \( v \in V \), we have \((c_1 + c_2)v = c_1v + c_2v \).
8. For all \( c_1, c_2 \in F \) and for all \( v \in V \), we have \((c_1c_2)v = (c_1c_2)v \).

The list of axioms is again boring but easy to remember. If we regard vectors as arrows in the plane, or as forces, then these axioms state obvious compatibility relationships. The reader should check that, in the plane, the definitions
\[
(x, y) + (u, v) = (x + u, y + v)
\]
\[
c(x, y) = (cx, cy)
\]
correspond to the usual geometric meanings of addition and scalar multiplication.

**Remark.** The axioms in Definition 5.2 imply additional basic facts, most of which we use automatically. For example, for all \( v \in V \), we have \( 0v = 0 \). To illustrate proof techniques, we pause to verify this conclusion. Some readers might find the proof pedantic and dull; others might enjoy the tight logic used.

Since \( 0 + 0 = 0 \) in a field, we have \( 0v = (0 + 0)v \). By axiom (7), we get \( 0v = (0 + 0)v = 0v + 0v \). By axiom (4), there is a vector \(-(0v)\) with \( 0 = -(0v) + 0v \). We add this vector to both sides and use axioms (3) and (1):
\[
0 = -(0v) + 0v = -(0v) + (0v + 0v) = (-0v + 0v) + 0v = 0 + 0v = 0v.
\]

We have replaced the standard but vague phrase “a vector is an object with both magnitude and direction” with “a vector is an element of a vector space”. Later on we will consider situations in which we can measure the length (or norm, or magnitude) of a vector.

The diversity of examples below illustrates the power of abstraction. The ideas of linearity apply in all these cases.

**Example 5.1 (vector spaces).** The following objects are vector spaces:

1. \( \mathbb{R}^n \) with addition and scalar multiplication defined as in (7.1) and (7.2).
2. \( \mathbb{C}^n \) with addition and scalar multiplication defined as in (7.1) and (7.2).
Let $S$ be any set and let $V_S = \{\text{functions } f : S \to \mathbb{R}\}$. Then $V_S$ is a vector space. The zero vector is the function that is identically 0 on $S$. Here we define addition and scalar multiplication by
\[(f + g)(s) = f(s) + g(s) \quad \text{and} \quad (cf)(s) = cf(s).\] (8)

The set of real- or complex-valued continuous functions on a subset $S$ of $\mathbb{R}^n$. Addition and scalar multiplication are defined as in (8).

The solutions to a linear homogeneous differential equation.

The collection of states in quantum mechanics.

For any vector space $V$, the collection of linear maps from $V$ to the scalar field is a vector space, called the dual space of $V$.

We mention another example with a rather different feeling. Let $F$ be the field consisting only of the two elements 0 and 1 with $1 + 1 = 0$. Exercises 5.6-10 give interesting uses of this field. Consider $n$-tuples consisting of zeroes and ones. We add componentwise and put $1 + 1 = 0$. This vector space $F^n$ and its analogues for prime numbers larger than 2 arise in computer science and cryptography.

Example 5.2. Consider the lights-out puzzle; it consists of a five-by-five array of lights. Each light can be on or off. Pressing a light toggles the light itself and its neighbors, where its neighbors are those locations immediately above, to the right, below, and to the left. For example, pressing the entry $P_{11}$ toggles the entries $P_{11}, P_{12}, P_{21}$. (There is no neighbor above and no neighbor to the left.) Pressing the entry $P_{33}$ toggles $P_{33}, P_{32}, P_{34}, P_{43}, P_{32}$. We can regard a configuration of lights as a point in a 25 dimensional vector space over the field of two elements. Toggling amounts to adding 1, where addition is taken modulo 2. Associated with each point $P_{ij}$, there is an operation $A_{ij}$ of adding a certain vector of ones and zeroes to the array of lights. For example, $A_{13}$ adds one to the entries labeled $P_{23}, P_{33}, P_{43}, P_{32}, P_{34}$ while leaving the other entries alone.

The following example from calculus shows one way in which linear algebra in infinite dimensions differs from linear algebra in finite dimensions.

Example 5.3. Let $V$ denote the space of polynomials in one real variable. Thus $p \in V$ means we can write $p(x) = \sum_{j=0}^{N} a_j x^j$. Define $D : V \to V$ by $D(p) = p'$ and $J : V \to V$ by $J(p)(x) = \int_{0}^{x} p(t)dt$. Both $D$ and $J$ are linear. Moreover, $DJ = I$ since
\[(DJ)(p)(x) = \frac{d}{dx} \int_{0}^{x} p(t)dt = p(x).\]
Therefore $DJ(p) = p$ for all $p$ and hence $DJ = I$. By contrast,
\[(JD)(p)(x) = \int_{0}^{x} p'(t)dt = p(x) - p(0).\]
Hence, if $p(0) \neq 0$, then $(JD)p \neq p$. Therefore $JD \neq I$.

In general, having a one-sided inverse does not imply having an inverse. In finite dimensions, however, a one-sided inverse is also a two-sided inverse.

In the previous section we showed how to use row operations to find the inverse of (an invertible) square matrix. There is a formula, in terms of the matrix entries, for the inverse. Except for two-by-two matrices, the formula is too complicated to be of much use in computations.

Example 5.4. Suppose that $L : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by
\[L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + y \end{pmatrix}.\]
Then \( L \) is invertible, and the inverse mapping \( L^{-1} \) is matrix multiplication by
\[
\begin{pmatrix}
1 & -1 \\
-1 & 2
\end{pmatrix}.
\]
A general 2-by-2 matrix
\[
L = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]
has an inverse if and only if \( ad - bc \neq 0 \). The number \( ad - bc \) is called the determinant of \( L \). See Section 9. When \( ad - bc \neq 0 \), we have
\[
L^{-1} = \frac{1}{ad - bc} \begin{pmatrix}
d & -b \\
-c & d
\end{pmatrix}.
\]

Suppose \( U \subseteq V \) and \( V \) is a vector space. If \( U \) is also a vector space under the same operations, then \( U \) is called a \textbf{subspace} of \( V \). Many of the vector spaces arising in this book are naturally given as subspaces of known vector spaces. To check whether \( U \) is a subspace of \( V \), one simply must check that \( U \) is closed under both operations. In particular, \( U \) must contain the zero vector.

\textbf{Example 5.5.} Suppose \( L : V \to W \) is linear. Then
\[
N(L) = \{ v : Lv = 0 \},
\]
the null space of \( L \), is a vector subspace of \( V \). Also the range of \( L \),
\[
R(L) = \{ u : u = Lv, \text{ for some } v \in V \},
\]
is a vector subspace of \( W \). The proof is simple: if \( w_1 = Lv_1 \) and \( w_2 = Lv_2 \), then (by the linearity of \( L \))
\[
w_1 + w_2 = Lv_1 + Lv_2 = L(v_1 + v_2).
\]
If \( w = Lv \), then (by the other aspect of linearity), \( cw = c(Lv) = L(cv) \). The range is therefore closed under both operations and hence it is a subspace of \( W \).

\textbf{Example 5.6.} Eigenspaces (defined formally in Definition 7.1) are important examples of subspaces. Assume \( L : V \to V \) is linear. Let
\[
E_\lambda(L) = \{ v : Lv = \lambda v \}
\]
denote the eigenspace corresponding to \( \lambda \). It is a subspace of \( V \): if \( L(v_1) = \lambda v_1 \) and \( L(v_2) = \lambda v_2 \), then
\[
L(v_1 + v_2) = Lv_1 + Lv_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2),
\]
\[
L(cv) = cL(v) = c\lambda v = \lambda(cv).
\]
Thus \( E_\lambda \) is closed under both sum and scalar multiplication.

\textbf{Example 5.7.} Let \( V \) be the space of continuous functions on the interval \([0, 1]\). The set of functions that vanish at a given point \( p \) is a subspace of \( V \). The set of functions that are differentiable on the open interval \((0, 1)\) is also a subspace of \( V \).

\textbf{Example 5.8.} A line in the plane \( \mathbb{R}^2 \) is a subspace if and only if it contains \((0, 0)\). If the line does not go through \((0, 0)\), then the set of points on the line is not closed under scalar multiplication by 0. If the line does go through \((0, 0)\), then we parametrize the line by \((x, y) = t(a, b) \) for \( t \in \mathbb{R} \). Here \((a, b)\) is a non-zero vector giving the direction of the line. Closure under addition and scalar multiplication is then evident. The map \( t \to (at, bt) \) is linear from \( \mathbb{R} \) to \( \mathbb{R}^2 \).

We close this section by solving an exercise in abstract linear algebra. Our purpose is to help the reader develop skill in abstract reasoning.
EXAMPLE 5.9. Suppose $L : V \to V$ is linear. Consider the following two properties:

1. $L(Lv) = 0$ implies $Lv = 0$. 
2. $\mathcal{N}(L) \cap \mathcal{R}(L) = \{0\}$.

Prove that (1) holds if and only if (2) holds.

**Proof.** We must do two things. Assuming (1), we must verify (2). Assuming (2), we must verify (1).

First we assume (1). To establish (2) we pick an arbitrary element $w$ in both the null space and the range. We must somehow use (1) to show that $w = 0$. We are given $Lw = 0$ and that $w = Lv$ for some $v$. Therefore $0 = Lw = L(Lv)$. By the hypothesis in (1), we conclude that $Lv = 0$. But $w = Lv$ and hence $w = 0$. We have established (2).

Second we assume (2). To establish (1) we must assume $L(Lv) = 0$ and somehow show that $Lv = 0$. But $Lv$ is in the range of $L$ and, if $L(Lv) = 0$, then $Lv$ is also in the null space of $L$. Therefore $Lv$ is in $\mathcal{N}(L) \cap \mathcal{R}(L)$. By (2), we have $Lv = 0$. We have established (1). \[\]

**Exercise 5.1.** For $d > 0$, the set of polynomials of degree $d$ is not a vector space. Why not?

**Exercise 5.2.** Show that the set of $m$-by-$n$ matrices with real entries is a vector space.

**Exercise 5.3.** Show that the set of functions $f : \mathbb{R} \to \mathbb{R}$ with $f(1) = 0$ is a vector space. Show that the set of functions $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 1$ is not a vector space.

**Exercise 5.4.** Is the set $\mathbb{R}$ of real numbers a vector space over the field $\mathbb{Q}$ of rational numbers?

**Exercise 5.5.** The set of square matrices $A$ with real entries is a vector space. (See Exercise 5.2.) Show that the collection of $A$ for which $\sum a_{jj} = 0$ is a subspace.

The next several exercises illustrate to some extent how linear algebra can change if we work over a field other than the real or complex numbers.

**Exercise 5.6.** Consider the matrix

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$ 

Find the null space of $A$ assuming the scalars are real numbers. Find a basis for the null space of $A$ assuming the scalars come from the field $\mathbb{F}$ of two elements. ($1 + 1 = 0$ in this field.)

**Exercise 5.7.** Again suppose that $\mathbb{F}$ is the field of two elements. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

Show that $A^{-1} = A$. Find all the two-by-two matrices $A$ with real entries and $A^{-1} = A$.

**Exercise 5.8.** This problem is useful in analyzing the lights-out puzzle from Example 5.2. Consider the five-by-five matrix

$$L = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$ 

Determine the null space of $L$, assuming first that the scalar field is the real numbers. Next determine the null space of $L$, assuming that the scalar field is the field of two elements.
The next exercise has an interesting consequence. In computer science the notion of exclusive or is more natural than the notion of union. Intersection corresponds to multiplication in the field of two elements; exclusive or corresponds to addition in this field.

**Exercise 5.9.** Let 0 denote the concept of false and 1 denote the concept of true. Given logical statements \( P \) and \( Q \), we assign 0 or 1 to each of them in this way. Verify the following:

- The statement \( P \) and \( Q \) both hold, written \( P \cap Q \), becomes \( PQ \mod (2) \).
- The statement that \( P \) or \( Q \) holds but not both, written \( P \oplus Q \), becomes \( P + Q \mod (2) \).

**Exercise 5.10.** Let \( F \) be the field of two elements. Show that the function \( f \) given by \( f(x) = 1 + x + x^2 \) is a constant.

### 6. Dimension

We have discussed dimension in an informal way. In this section we formalize the notion of dimension of a vector space, using the fundamental concepts of linear independence and span. We begin with an important pedagogical remark.

**Remark.** Beginning students sometimes wonder why mathematicians consider dimensions higher than three. One answer is that higher dimensions arise in many applied situations. Consider for example the position \( r(t) \) at time \( t \) of a rectangular eraser thrown into the air by the professor. It takes six numbers to specify \( r(t) \). It takes another six to describe its velocity vector. For another example, consider modeling baseball using probability. In a given situation, there are eight possible ways for baserunners to be located and three possibilities for the number of outs. Ignoring the score of the game, twenty-four parameters are required to specify the situation. One could also take balls and strikes into account and multiply the number of parameters by twelve. In general, dimension tells us how many independent pieces of information are needed to describe a given situation.

The dimension of a vector space tells us how many independent parameters are needed to specify each vector. To define dimension, we need to understand the concept of a basis, which is defined formally in Definition 6.4. If a vector space \( V \) has a finite basis, then we define the dimension of \( V \) to be the number of elements in this basis. For this definition to be valid we must check that this number does not depend upon the basis chosen. See Corollary 6.6. By convention, the vector space consisting of the zero vector alone has dimension 0. When no finite basis exists and the vector space does not consist of the zero vector alone, we say that the vector space has infinite dimension. We write \( \dim(V) \) for the dimension of a vector space \( V \). Soon it will become clear that \( \dim(\mathbb{R}^n) = n \).

In the Kakuro example from Section 1, the null space had dimension 2, and the range had dimension 5. The sum is 7, which is the dimension of the domain, or the number of variables. The following general result is easily proved by row operations. We sketch its proof to illustrate the ideas that follow. We also illustrate the result with examples.

**Theorem 6.1.** Assume \( L : \mathbb{R}^n \to \mathbb{R}^k \) is linear. Then

\[
\dim \mathcal{N}(L) + \dim \mathcal{R}(L) = n.
\]

**Proof.** (Sketch) After bases are chosen, we may assume that \( L \) is defined via matrix multiplication. Note that \( n \) is the number of columns in this matrix. Each row operation preserves \( \mathcal{N}(L) \) and \( \mathcal{R}(L) \). When we reach row echelon form, the number of leading ones is the number of independent columns, which is the dimension of the range. The dimension of the null space is the number of arbitrary variables in the general solution, and hence is \( n \) minus the number of independent columns. In other words, the result is
obvious when a matrix is in row-echelon form. Row operations preserve \( N(L) \) and \( R(L) \) and hence their dimensions. The result follows.

**Example 6.1.** Define \( L : \mathbb{R}^3 \to \mathbb{R}^3 \), where \( L \) is matrix multiplication by

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & -1 \\
1 & 0 & 1
\end{pmatrix}
\]

Put \( b = (31, 6, 7) \). To solve \( Lx = b \), we use row operations as usual. The system is the matrix equation:

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & -1 \\
1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = \begin{pmatrix}
31 \\
6 \\
7
\end{pmatrix}.
\]

We write the system as an augmented matrix and perform two row operations:

\[
\begin{pmatrix}
1 & 2 & 3 & 31 \\
0 & 1 & -1 & 6 \\
1 & 0 & 1 & 7
\end{pmatrix} \to \begin{pmatrix}
1 & 2 & 3 & 31 \\
0 & 1 & -1 & 6 \\
0 & -2 & -2 & -12
\end{pmatrix}
\]

We obtain \( x = (4, 9, 3) \). In this example, \( L \) is invertible and hence \( N(L) = \{0\} \). Also \( R(L) = \mathbb{R}^3 \).

**Example 6.2.** In this example the linear map is not invertible. Take

\[
M = \begin{pmatrix}
1 & 2 & 3 \\
0 & 1 & -1 \\
1 & 3 & -2
\end{pmatrix}
\]

We use row operations to find the null space and range of the linear mapping corresponding to the matrix \( M \). Assume \((a, b, c)\) is in the range. We try to solve:

\[
\begin{pmatrix}
1 & 2 & 3 & a \\
0 & 1 & -1 & b \\
1 & 3 & 2 & c
\end{pmatrix} \to \begin{pmatrix}
1 & 2 & 3 & a \\
0 & 1 & -1 & b \\
0 & 0 & 0 & c - b - a
\end{pmatrix} \to \begin{pmatrix}
1 & 0 & 5 & a - 2b \\
0 & 1 & -1 & b \\
0 & 0 & 0 & c - b - a
\end{pmatrix}.
\]

For a solution to exist, we require \( c - a - b = 0 \). Conversely, when this condition is satisfied we solve the system by letting \( x_3 \) be arbitrary, putting \( x_2 = b + x_3 \), and putting \( x_1 = a - 2b - 5x_3 \). Therefore,

\[
R(M) = \{ \begin{pmatrix} a \\ b \\ a + b \end{pmatrix} \} = \{ a \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \}.
\]

The range of \( M \) thus has dimension 2. The null space of \( M \) has dimension 1; it consists of all multiples of \((-5, 1, 1)\). Notice that \( \dim(N(M)) + \dim(R(M)) = 1 + 2 = 3 \) here, illustrating Theorem 6.1.

We formally introduce the basic concepts of linear independence and span.

**Definition 6.2.** A subset \( \{v_1, \ldots, v_l\} \) of a vector space is **linearly independent** if

\[
\sum_{j=0}^l c_j v_j = 0 \Rightarrow c_j = 0 \text{ for all } j.
\]

The reader should be familiar with this concept from ODE.
Example 6.3. Consider the ODE $L(y) = y'' - (\alpha + \beta)y' + \alpha \beta y = 0$. We regard $L$ as a linear mapping defined on the space of twice differentiable functions on $\mathbb{R}$ or $\mathbb{C}$. Solving $Ly = 0$ is the same as finding the null space of $L$. We have

$$\mathcal{N}(L) = \begin{cases} 
  c_1 e^{\alpha x} + c_2 e^{\beta x} & \text{if } \alpha \neq \beta \\
  c_1 e^{\alpha x} + c_2 x e^{\alpha x} & \text{if } \alpha = \beta 
\end{cases}$$

To show that $e^{\alpha x}$ and $e^{\beta x}$ are independent for $\alpha \neq \beta$, consider the equations

$$\begin{cases} 
  c_1 e^{\alpha x} + c_2 e^{\beta x} = 0 \\
  c_1 \alpha e^{\alpha x} + c_2 \beta e^{\beta x} = 0.
\end{cases}$$

The second equation is obtained by differentiating the first. Rewrite these equations as a matrix equation:

$$\begin{pmatrix} 
  e^{\alpha x} & e^{\beta x} \\
  \alpha e^{\alpha x} & \beta e^{\beta x}
\end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

We claim that the null space of

$$\begin{pmatrix} 
  e^{\alpha x} & e^{\beta x} \\
  \alpha e^{\alpha x} & \beta e^{\beta x}
\end{pmatrix}$$

is the zero vector $\{0\}$. Since $e^{\alpha x} \neq 0$ and $e^{\beta x} \neq 0$, this statement holds if and only if

$$\begin{pmatrix} 1 & 1 \\
  \alpha & \beta 
\end{pmatrix}$$

is invertible, which holds if and only if $\beta - \alpha \neq 0$. When $\alpha \neq \beta$, the functions $e^{\alpha x}$ and $e^{\beta x}$ are therefore linearly independent (also see Exercise 6.1). In case $\alpha = \beta$, one checks that the null space is spanned by $e^{\alpha x}$ and $x e^{\alpha x}$. These functions are linearly independent because

$$0 = c_1 e^{\alpha x} + c_2 x e^{\alpha x}$$

implies $0 = c_1 + c_2 x$ for all $x$, and hence $c_1 = c_2 = 0$.

**Definition 6.3.** A collection of vectors $v_1, \ldots, v_N$ spans a vector space $V$ if, for each vector $v \in V$ there are scalars $c_1, \ldots, c_N$ such that

$$v = \sum_{j=1}^{N} c_j v_j.$$  

For example, in Example 6.2, we found two independent vectors that span $\mathcal{R}(M)$. We do not define span in the infinite-dimensional case at this time.

**Definition 6.4.** A collection of vectors $v_1, \ldots, v_N$ is a basis for a vector space $V$ if this collection both spans $V$ and is linearly independent.

We observe a simple fact relating linear independence and span.

**Proposition 6.5.** Let $U$ be a subspace of $V$. If $u_1, u_2, \ldots, u_k$ span $U$, and $v_1, \ldots, v_n$ are linearly independent elements of $U$, then $k \geq n$. In particular,

$$n \leq \dim(U) \leq k.$$  

**Proof.** We sketch the idea. Given a spanning set $u_1, \ldots, u_k$ for $U$ and a linearly independent set $v_1, \ldots, v_n$ in $U$, we inductively exchange $n$ of the $u_j$ for the $v_j$. After renumbering the $u_j$’s, we get a spanning set $v_1, \ldots, v_n, u_{n+1}, \ldots, u_k$.  

**Corollary 6.6.** Each basis of (a finite-dimensional) vector space has the same number of elements.
PROOF. Suppose both \( w_1, \ldots, w_k \) and \( u_1, \ldots, u_n \) are bases. Since the \( u_j \) are linearly independent and the \( w_j \) span, \( n \leq k \). Since the \( w_j \) are linearly independent and the \( u_j \) span, \( k \leq n \). Hence \( k = n \). \( \square \)

We summarize. The dimension of a vector space \( V \) equals the positive integer \( k \) if \( V \) has a basis with \( k \) elements. By Corollary 6.6, the dimension is well-defined. The dimension of the vector space consisting of the origin alone has dimension 0; there are no linearly independent elements and hence no basis. The dimension is infinite otherwise.

Exercise 6.6 asks for a proof of a general dimension formula. This result precisely expresses ideas about degrees of freedom, interdependence of variables, and related ideas. To help prepare for that exercise, and to illustrate the basic ideas, we sketch the solution of a simple exercise in the next example.

Example 6.4. The set \( W_1 \) of vectors of the form \((x, -x, y, z)\) is a subspace of \( \mathbb{R}^4 \), and the set \( W_2 \) of vectors of the form \((a, b, -a, c)\) is also a subspace. (The reader who doesn’t see why should re-read the definition of subspace!) Verify the formula from Exercise 4.6 in this case:

\[
\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2).
\]

Recall that the dimension of a space is the number of elements in a basis. A basis for \( W_1 \) is given by the three vectors

\[
(1, -1, 0, 0) \quad (0, 0, 1, 0) \quad (0, 0, 0, 1).
\]

A basis for \( W_2 \) is given by the three elements

\[
(1, 0, -1, 0) \quad (0, 1, 0, 0) \quad (0, 0, 0, 1).
\]

The intersection \( W_1 \cap W_2 \) is the set of vectors of the form \((x, -x, -x, c)\). A basis for \( W_1 \cap W_2 \) is therefore given by the two vectors

\[
(1, -1, -1, 0) \quad (0, 0, 0, 1).
\]

Every vector in \( \mathbb{R}^4 \) is the sum of a vector in \( W_1 \) and \( W_2 \). Therefore \( \dim(W_1 + W_2) = 4 \). In this case the formula therefore reads \( 3 + 3 = 2 + 4 \).

Exercise 6.1. Let \( V \) be the vector space of functions on \( \mathbb{R} \). In each case show that the functions are linearly independent:

- \( 1, x, x^2, \ldots, x^n \).
- \( p_1(x), \ldots, p_n(x) \) if each \( p_j \) is a polynomial of different degree.
- \( e^{a_1 x}, e^{a_2 x}, \ldots, e^{a_n x} \). Assume that the \( a_j \) are distinct.
- \( x \) and \( |x| \).

Exercise 6.2. Let \( a_j \) be \( n \) distinct real numbers. Define polynomials \( p_j \) by

\[
p_j(x) = \prod_{j \neq i}(x - a_i).
\]

Show that these polynomials are linearly independent.

Exercise 6.3. Let \( V \) be the vector space of all functions \( f : \mathbb{R} \to \mathbb{R} \). Given \( f, g \in V \), show that \( \min(f, g) \) is in the span of \( f + g \) and \( |f - g| \). Determine an analogous result for the maximum.

Exercise 6.4. Let \( V \) be the vector space of all functions \( f : \mathbb{R} \to \mathbb{R} \). True or false? \( |x| \) is in the span of \( x \) and \( -x \).

Exercise 6.5. Determine whether \((6, 4, 2)\) is in the span of \((1, 2, 0)\) and \((2, 0, 1)\). Determine for which values of \( c \) the vector \((4, 7, c)\) is in the span of \((1, 2, 3)\) and \((2, 3, 4)\).
Exercise 6.6. Let $U$ and $V$ be subspaces of a finite-dimensional vector space $W$. Follow the following outline to prove that

$$\dim(U + V) + \dim(U \cap V) = \dim(U) + \dim(V).$$

Here $U + V$ is the span of $U$ and $V$. First find a basis $\{\alpha_j\}$ for $U \cap V$. Extend it a basis $\{\alpha_j, \beta_j\}$ for $U$ and to a basis $\{\alpha_j, \gamma_j\}$ for $V$. Then show that the vectors $\{\alpha_j, \beta_j, \gamma_j\}$ are linearly independent and span $U + V$.

Exercise 6.7. Fill in the details from Example 6.4.

Exercise 6.8. Use the language of linear algebra to explain the following situation. A circuit has $k$ unknown currents and many known voltages and impedances. Kirchhoff’s laws yield more than $k$ equations. Yet there is a unique solution for the list of currents.

Exercise 6.9. Give an example of a linear map $L$ for which both $\mathcal{N}(L)$ and $\mathcal{R}(L)$ are positive dimensional and also $\mathcal{N}(L) \cap \mathcal{R}(L) = \{0\}$.

Exercise 6.10. Find an analogue of the formula from Example 6.4 for three subspaces $W_1$, $W_2$, and $W_3$. Prove your formula. Suggestion: Consider the inclusion-exclusion principle.

7. Eigenvalues and eigenvectors

Definition 7.1. Let $V$ be a real or complex vector space. Suppose $L : V \to V$ is linear. A scalar $\lambda$ is called an eigenvalue of $L$ if there is a non-zero vector $v$ such that $Lv = \lambda v$. Such a vector $v$ is called an eigenvector of $L$ and $\lambda$ is called the eigenvalue corresponding to the eigenvector $v$. When $\lambda$ is an eigenvalue of $L$, the set

$$E_\lambda = \{v : Lv = \lambda v\}$$

is a subspace of $V$, called the eigenspace corresponding to $\lambda$.

Eigenvalues are also known as spectral values and arise throughout science; the Greek letter $\lambda$ suggests wave length. We will provide many examples and explanations as we continue.

Example 7.1. Why is the exponential function so important? One of several related reasons is that the function $x \mapsto e^{\lambda x} = f(x)$ has the following property: $Df = \lambda f$. Thus $f$ is an eigenvector, with eigenvalue $\lambda$, for the linear operator known as differentiation. We will return to this idea in our section on Fourier series in Chapter 5.

In most of this section we work in finite-dimensional spaces. Our linear maps will be given by square matrices. It is nonetheless important to understand that many problems in applied mathematics involve infinite-dimensional versions of the same ideas.

Definition 7.2. A linear transformation $L : V \to V$ is called diagonalizable if $V$ has a basis consisting of eigenvectors of $L$.

When $L$ is diagonalizable, with $v_1, \ldots, v_N$ a basis of eigenvectors, the matrix of $L$ with respect to this basis (in both the domain and target copies of $V$) is diagonal. The diagonal elements are the corresponding eigenvalues. We will see an infinite-dimensional analogue when we study the Sturm-Liouville differential equation in Chapter 6.

In Example 7.2 we compute eigenspaces in a simple situation to illustrate the useful notion of diagonalization. A matrix $D_{ij}$ is diagonal when $D_{ij} = 0$ for $i \neq j$. Computing with diagonal matrices is easy. A matrix $A$ is called diagonalizable if there is an invertible matrix $P$ and a diagonal $D$ such that $A = PDP^{-1}$. We will see that this condition means that $A$ behaves like a diagonal matrix, when we choose the correct basis. When $A$ is regarded as a linear transformation, the column vectors of $P$ are the eigenvectors of $A$. 
The two uses of the term diagonalizable (for a matrix and for a linear map) are essentially the same. Consider a diagonal matrix $D$, with elements $\lambda_1, \ldots, \lambda_n$ on the diagonal. Let $e_j$ denote the vector which is 1 in the $j$-th slot and 0 in the other slots. For each $j$, we have $D(e_j) = \lambda_j e_j$. Hence $e_j$ is an eigenvector of $D$ with corresponding eigenvalue $\lambda_j$. Consider the matrix $A = PDP^{-1}$. Then

$$A(P(e_j)) = (PDP^{-1})(Pe_j) = PD(e_j) = P(\lambda_j e_j) = \lambda_j Pe_j.$$ 

Therefore $P(e_j)$ is an eigenvector for $A$ with corresponding eigenvalue $\lambda_j$. A linear map is diagonalizable if and only if its matrix, with respect to a basis of eigenvalues, is diagonal.

**Example 7.2.** Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by the matrix

$$L = \begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix}. \quad (11)$$

We find the eigenvalues $\lambda$ and the corresponding eigenvectors. Setting $v = (x, y)$ and $Lv = \lambda v$ yields

$$\begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}.$$ 

Hence

$$\begin{pmatrix} -1 - \lambda & 10 \\ -5 & 14 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$ 

Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a non-trivial null space if and only if $ad - bc = 0$, we obtain the characteristic equation

$$0 = (\lambda + 1)(\lambda - 14) + 50 = \lambda^2 - 13\lambda + 36 = (\lambda - 4)(\lambda - 9).$$

Thus $\lambda_1 = 4$ and $\lambda_2 = 9$ are the eigenvalues of $\begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix}$. To find the corresponding eigenvectors, we must find the null spaces of $L - 4I$ and $L - 9I$. If $Lv = 4v$, we find that $v$ is a multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Also $Lv = 9v$ implies that $v$ is a multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. We can check the result of our calculations:

$$\begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$ 

$$\begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix} = 9 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$ 

When the eigenvectors of a linear transformation $L : V \to V$ form a basis of $V$, we can diagonalize $L$. In this case we write $L = PDP^{-1}$, where the columns of $P$ are the eigenvectors of $L$, and $D$ is diagonal. See Figures 6 and 7 for the geometric interpretation of diagonalization of the matrix $L$ from (11).

$$L = \begin{pmatrix} -1 & 10 \\ -5 & 14 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = PDP^{-1}. \quad (12)$$

**Remark.** For a two-by-two matrix $L$, there is a simple way to check whether $L$ is diagonalizable and if so, to diagonalize it. A two-by-two matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has two invariants, its determinant $ad - bc$ and its trace $a + d$. The eigenvalues $\lambda_1$ and $\lambda_2$ must satisfy the system of equations

$$\lambda_1 + \lambda_2 = \text{trace}(L) = a + d$$ 

$$\lambda_1 \lambda_2 = \text{det} L = ad - bc.$$
If this (quadratic) system has two distinct solutions, then the matrix is diagonalizable. If the roots are repeated, then the matrix is not diagonalizable unless it is already diagonal. See also Exercises 7.4 and 7.5.

**Example 7.3.** We use the remark to verify (12). The trace of $L$ is 13 and the determinant of $L$ is 36. We have $4 + 9 = 13$ and $4 \cdot 9 = 36$. Hence 4 and 9 are the eigenvalues. Since $L \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 4. Similarly $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the eigenvalue 9. Therefore (12) holds.

\[ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \]

\[ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

**Figure 6.** A linear map using one basis

\[ L : \mathbb{R}^2 \to \mathbb{R}^2 \]

\[ D : \mathbb{R}^2 \to \mathbb{R}^2 \]

**Figure 7.** The same linear map using eigenvectors as a basis
REMARK. Not every square matrix with real entries has real eigenvalues. The following example both explains why and helps anticipate our work on complex numbers. Later in this book we will consider unitary and Hermitian matrices. Unitary matrices will have their eigenvalues on the unit circle and Hermitian matrices will have only real eigenvalues. Both circumstances have significant applications in physics and engineering.

**Example 7.4.** Consider the two-by-two matrix

$$J_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

as a mapping from $\mathbb{R}^2$ to itself. The geometric interpretation of $J_\theta$ is a counterclockwise rotation by the angle $\theta$. Unless $\theta$ is a multiple of $\pi$, such a transformation cannot map a non-zero vector to a multiple of itself. Hence $J_\theta$ will have no real eigenvalues. The special case when $\theta = \frac{\pi}{2}$ corresponds to multiplication by the complex number $i$.

Formula (12) enables us to compute functions of $L$. Finding functions of a linear operator will be a recurring theme in this book. To anticipate the ideas, consider a polynomial $f(x) = \sum_{j=0}^{d} c_j x^j$. We can substitute $L$ for $x$ and consider $f(L)$. We can find $f(L)$ using (12). When $D$ is diagonal, it is easy to compute $f(D)$. See also the section on similarity at the end of this chapter for an analogue of (12) and an application. The reader who has heard of Laplace or Fourier transforms should think about the sense in which these transforms diagonalize differentiation. This idea enables one to make sense of fractional derivatives.

**Remark.** Assume $V$ is finite-dimensional and $L : V \to V$ is linear. The collection of eigenvalues of $L$ is known as the spectrum of $L$ and is often denoted by $\sigma(L)$. In the finite-dimensional case, $\lambda$ is an eigenvalue if and only if $L - \lambda I$ is not invertible. In the infinite-dimensional case, the spectrum consists of those scalars $\lambda$ for which $(L - \lambda I)^{-1}$ does not exist. See [RS] for discussion about this distinction. In this book, we will encounter continuous spectrum only briefly, in the last chapter.

**Exercise 7.1.** With $L$ as in (11), compute $L^2 - 3L$ in two ways. First find it directly. Then find it using (12). Use (12) to find (all possibilities for) $p L$.

**Exercise 7.2.** Find the eigenvectors and eigenvalues of the matrix $\begin{pmatrix} 20 & -48 \\ 8 & -20 \end{pmatrix}$.

**Exercise 7.3.** Compute the matrix product $P^{-1}QP$, and easily check your answer, if

$$Q = \begin{pmatrix} -1 & 8 \\ -4 & 11 \end{pmatrix}$$

$$P = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

**Exercise 7.4.** Show that $\begin{pmatrix} 1 & 4 \\ -1 & 5 \end{pmatrix}$ is not diagonalizable.

**Exercise 7.5.** Determine a necessary and sufficient condition (in terms of the entries) for a 2-by-2 matrix to be diagonalizable.

**Exercise 7.6.** Let $V$ be the space of continuous real-valued functions on $\mathbb{R}$. Let $W$ be the subspace for which the integral below converges. Define a linear map $L : W \to V$ by

$$(Lf)(x) = \int_{-\infty}^{x} f(t)dt.$$
Determine all eigenvalues and eigenvectors of \( L \). Do the same problem if

\[
(Lf)(x) = \int_{-\infty}^{x} tf(t)dt.
\]

In both cases, what happens if we replace the lower limit of integration with 0?

**Exercise 7.7.** Let \( V \) be the space of infinitely differentiable complex-valued functions on \( \mathbb{R} \). Let \( D : V \to V \) denote differentiation. What are the eigenvalues of \( D \)?

**Exercise 7.8.** Find the complex eigenvalues of the matrix \( J_{\theta} \) from Example 7.4.

**Exercise 7.9.** True or false? A 2-by-2 matrix of complex numbers always has a square root.

### 8. Bases and the matrix of a linear map

We often express linear transformations in finite dimensions as matrices. We emphasize that the matrix depends on a choice of basis. The ideas here are subtle. We begin by discussing different vector spaces of the same dimension and different bases of the same vector space.

Given vector spaces of the same dimension, to what extent can we regard them as the same? Bases allow us to do so, as follows. Vector spaces \( V, W \) (over the same field) are **isomorphic** if there is a linear map \( L : V \to W \) such that \( L \) is both injective (one-to-one) and surjective (onto). The map \( L \) is called an **isomorphism**. It is routine to check that isomorphism is an equivalence relation.

**Remark.** Assume \( A \) and \( B \) are sets and \( f : A \to B \) is a function. Then \( f \) is called **injective** or **one-to-one** if \( f(x) = f(y) \) implies \( x = y \). Also \( f \) is called **surjective** or **onto** if, for each \( b \in B \), there is an \( a \in A \) such that \( f(a) = b \). We can think of these words in the context of solving an equation. To say that \( f \) is surjective means for each \( b \) that we can solve \( f(a) = b \). To say that \( f \) is injective means that the solution, if it exists, is unique. When both properties hold, we get both existence and uniqueness. Thus \( f \) is an isomorphism between \( A \) and \( B \): to each \( b \in B \) there is a unique \( a \in A \) for which \( f(a) = b \).

**Remark.** In the context of linear algebra, in order for a map \( L : V \to W \) to be an isomorphism, we also insist that \( L \) is linear.

Let us give a simple example. Consider the set \( V_3 \) of polynomials in one variable of degree at most 3. Then \( V_3 \) is a four-dimensional real vector space, spanned by the monomials \( 1, x, x^2, x^3 \). The space \( V_3 \) is not the same as \( \mathbb{R}^4 \), but there is a natural correspondence: put \( \phi(a + bx + cx^2 + dx^3) = (a, b, c, d) \). Then \( \phi : V_3 \to \mathbb{R}^4 \) is linear, injective, and surjective.

In the same manner, an ordered basis \( v_1, \ldots, v_n \) of \( V \) provides an isomorphism \( \phi : V \to \mathbb{R}^n \). When \( v = \sum c_j v_j \) we put \( \phi(v) = (c_1, \ldots, c_n) \). Since the inverse of an isomorphism is an isomorphism, and the composition of isomorphisms is itself an isomorphism, two vector spaces (with the same scalar field) of the same finite dimension are isomorphic.

Most readers should be familiar with the **standard** basis of \( \mathbb{R}^3 \). In physics one writes \( i, j, \) and \( k \) for the standard basis of \( \mathbb{R}^3 \). Other notations are also common:

\[
\begin{align*}
i &= (1, 0, 0) = e_1^* = e_1, \\
j &= (0, 1, 0) = e_2^* = e_2, \\
k &= (0, 0, 1) = e_3^* = e_3.
\end{align*}
\]

We can therefore write

\[
(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1) = ai + bj + ck.
\]
In $n$ dimensions, we use the notation $e_1, \ldots, e_n$ to denote the standard basis. Here $e_1 = (1, 0, \ldots, 0)$ and so on. Given $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ (or in $\mathbb{C}^n$), there are scalars $v_{jk}$ such that

$$v_1 = \sum_{j=1}^{n} v_{1j} e_j$$

$$v_2 = \sum_{j=1}^{n} v_{2j} e_j$$

$$v_n = \sum_{j=1}^{n} v_{nj} e_j.$$

The vectors $v_j$ themselves form a basis if and only if the matrix

$${\begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & \cdots & v_{nn} \end{bmatrix}}$$

is invertible. An ordered basis of an $n$-dimensional real vector space $V$ thus provides an isomorphism from $V$ to $\mathbb{R}^n$. Different ordered bases provide different isomorphisms.

We are finally ready to discuss the matrix of a linear map. Let $V$ and $W$ be finite-dimensional vector spaces. Assume $v_1, v_2, \ldots, v_n$ is a basis of $V$ and $w_1, w_2, \ldots, w_k$ is a basis of $W$. Suppose $L : V \rightarrow W$ is linear. What is the matrix of $L$ with respect to these bases?

For each index $l$, the vector $L(v_l)$ is an element of $W$. Since the $w_j$ are a basis for $W$, there are scalars such that we can write

$$L(v_1) = \sum_{j=1}^{k} c_{j1} w_j$$

$$L(v_2) = \sum_{j=1}^{k} c_{j2} w_j$$

$$\vdots$$

$$L(v_n) = \sum_{j=1}^{k} c_{jn} w_j.$$

Put these scalars into a matrix:

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kn} \end{pmatrix}.$$

Then $C$ is the matrix of $L$ with respect to these bases. The first column of $C$ consists of the scalars needed to write the image of the first basis vector of $V$ as a linear combination of the basis vectors of $W$. The same holds for each column. Thus the entries of $C$ depend on the bases and on the order the basis elements are selected.
Two points are worth making here. First, if we choose bases and find the matrix of a linear map \( L \) with respect to them, then we need to know what properties of \( L \) are independent of the bases chosen. For example, whether \( L \) is invertible is independent of these choices. Second, when we choose bases, we should choose them in a way that facilitates the computations. For example, when \( L : V \to V \) has a basis of eigenvectors, most computations are easiest when we choose this basis. See also Theorem 8.1 for an interesting example.

**Example 8.1.** Suppose \( L : \mathbb{R}^2 \to \mathbb{R}^2 \) is defined by \( L(x, y) = (x + y, x - y) \). The matrix of \( L \) with respect to the standard bases in both copies of \( \mathbb{R}^2 \) is

\[
\begin{pmatrix}
1 & 1 \\
1 & -1 \\
\end{pmatrix}
\]

Consider the basis \((1, 1)\) and \((0, 1)\) of the domain copy of \( \mathbb{R}^2 \) and the standard basis in the image copy of \( \mathbb{R}^2 \). The matrix of \( L \) with respect to these bases is now

\[
\begin{pmatrix}
2 & 1 \\
0 & -1 \\
\end{pmatrix}
\]

**Example 8.2.** Let \( V \) be the vector space of polynomials of degree at most \( N \). We define a linear map \( L : V \to V \) by

\[
Lp(x) = x \int_0^x p''(t) dt = x(p'(x) - p'(0)).
\]

Computing \( L(x^j) \) for \( 0 \leq j \leq N \) gives \( L(1) = L(x) = 0 \) and \( L(x^j) = jx^j \) otherwise. Each monomial is an eigenvector, and the matrix of \( L \) (with respect to the monomial basis) is diagonal.

The next example is more elaborate:

**Example 8.3.** Let \( V \) denote the space of polynomials of degree at most 3. Then \( \dim(V) = 4 \) and the polynomials \( 1, x, x^2, x^3 \) form a basis. Let \( D \) denote differentiation. Then \( D : V \to V \). The matrix of \( D \) with respect to the basis \( 1, x, x^2, x^3 \) in both the domain and target spaces is

\[
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

If instead we use the basis \( 1, x + 1, \frac{(x + 1)(x + 2)}{2}, \frac{(x + 1)(x + 2)(x + 3)}{6} \), then the matrix of \( D \) (with respect to this basis) is

\[
\begin{pmatrix}
0 & 1 & \frac{1}{2} & \frac{1}{3} \\
0 & 0 & 1 & \frac{1}{2} \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

The reader can check that

\[
D(1) = 0,
\]

\[
D(x + 1) = 1 + 0(x + 1) + 0\frac{(x + 1)(x + 2)}{2} + 0\frac{(x + 1)(x + 2)(x + 3)}{6}
\]

\[
D\left(\frac{(x + 1)(x + 2)}{2}\right) = x + \frac{3}{2} = \frac{1}{2} \cdot 1 + 1(x + 1)
\]

\[
D\left(\frac{(x + 1)(x + 2)(x + 3)}{6}\right) = \frac{1}{2}x^2 + 2x + \frac{11}{6} = \frac{1}{3} + \frac{1}{2}(x + 1) + \frac{(x + 1)(x + 2)}{2}.
\]
We now discuss an interesting situation where choosing the correct basis simplifies things. Consider polynomials with rational coefficients. Thus
\[ p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_kx^k, \]
where each \( a_j \in \mathbb{Q} \). What is the condition on these coefficients such that \( p(n) \) is an integer for every integer \( n \)? For example, \( p(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x \) has this property, even though its coefficients are not whole numbers.

Consider the combinatorial polynomials \( x + j \). For \( j = 0 \) this notation means the constant polynomial 1. For \( j \geq 1 \), the notation \( x + j \) means the polynomial \( (x+1)(x+2)...(x+j) \). These polynomials are of different degree and hence linearly independent over the rational numbers \( \mathbb{Q} \). By linear algebra, using \( \mathbb{Q} \) as the scalar field, we can write \( p(x) \) as a linear combination with rational coefficients of these polynomials. For example, we have
\[ \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x = 2\left(\frac{x+3)(x+2)(x+1)}{6}\right) - 3\left(\frac{(x+2)(x+1)}{2}\right) + \frac{x+1}{1}. \]
These combinatorial polynomials map the integers to the integers. Using them as a basis, we obtain the decisive answer.

**Theorem 8.1.** A polynomial with rational coefficients maps \( \mathbb{Z} \) to itself if and only if, when writing
\[ p(x) = \sum_{j=0}^{k} m_j \binom{x+j}{j}, \]
each \( m_j \) is an integer.

**Proof.** By Exercise 8.1, each combinatorial polynomial \( \binom{x+j}{j} \) maps the integers to the integers. Hence an integer combination of them also does, and the if direction follows. To prove the only if direction, consider a polynomial \( p \) of degree \( k \) with rational coefficients. There are rational numbers \( c_j \) such that
\[ p(x) = \sum_{j=0}^{k} c_j \binom{x+j}{j}, \]
because the polynomials \( \binom{x+j}{j} \) for \( 0 \leq j \leq k \) are linearly independent, and hence form a basis for the vector space of polynomials of degree at most \( k \) with rational coefficients. We must show that each \( c_j \) is an integer. To do so, first evaluate at \( x = -1 \). We get \( p(-1) = c_0 \). Since \( p(-1) \) is an integer, \( c_0 \) also is an integer. Then evaluate at \(-2 \). Since \( p(-2) \) is an integer, \( c_1 = p(-2) - c_0 \) is also an integer. Proceeding in this way we show inductively that all the coefficients are integers. See Exercise 8.3.

The next theorem provides another basis for the space \( \mathbb{Q}[x] \) of polynomials of degree at most \( d \) in one real (or complex) variable. Note that all the polynomials in the basis are of the same degree, and each appears in factored form.

**Theorem 8.2.** Let \( \lambda_1, \cdots, \lambda_n \) be distinct real numbers. For \( 1 \leq j \leq n \) define polynomials \( p_j(x) \) by
\[ p_j(x) = a_j \prod_{k \neq j}(x - \lambda_k), \]
9. Determinants

Determinants arise throughout mathematics, physics, and engineering. Many methods exist for computing them. It is interesting that row operations provide the most computationally useful method for finding determinants. At the end of this section we state a result summarizing much of our discussion on linear algebra; there we see the determinant as one of many tests for the invertibility of a matrix. The rest of the section reveals other uses for determinants.

Theorem 9.1. A linear map on a finite-dimensional space is invertible if and only if its determinant is non-zero.
As usual we will work with determinants of square matrices. Let $A$ be an $n$-by-$n$ matrix, whose elements are in $\mathbb{R}$ or $\mathbb{C}$. How do we define its determinant, $\det(A)$? We want $\det(A)$ to be a scalar that equals the oriented $n$-dimensional volume of the box spanned by the rows (or columns) of $A$. In particular, if $A$ is a diagonal matrix, then $\det(A)$ will be the product of the diagonal elements.

The key idea is to regard $\det$ as a function of the rows of a matrix. For this function to agree with our sense of oriented volume, it must satisfy certain properties. It should be linear in each row when the other rows are fixed. When the matrix fails to be invertible, the box degenerates, and the volume should be 0. To maintain the sense of orientation, the determinant should get multiplied by $-1$ when we interchange two rows. Finally, we normalize by saying that the identity matrix corresponds to the unit box, which has volume 1. These ideas uniquely determine the determinant function and tell us how to compute it.

We apply row operations to a given matrix. We find the determinant using the following properties. In (13.1) we interchange two rows:

$$\det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_l \end{pmatrix} = -\det \begin{pmatrix} r_1 \\ \vdots \\ r_l \\ \vdots \\ r_j \end{pmatrix} \tag{13.1}$$

In (13.2) we multiply a row by a constant:

$$\det \begin{pmatrix} r_1 \\ \vdots \\ cr_j \\ \vdots \\ r_l \end{pmatrix} = c \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_l \end{pmatrix} \tag{13.2}$$

In (13.3) we assume that $k \neq j$ and we add a multiple of the $k$-th row to the $j$-th row:

$$\det \begin{pmatrix} r_1 \\ \vdots \\ r_j + \lambda r_k \\ \vdots \\ r_l \end{pmatrix} = \det \begin{pmatrix} r_1 \\ \vdots \\ r_j \\ \vdots \\ r_l \end{pmatrix} \tag{13.3}$$

Finally we know the determinant of the identity matrix is 1:

$$\det(I) = 1. \tag{13.4}$$

**Example 9.1.**

$$\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 9 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -7 & -12 \end{pmatrix} = (-3)(-7) \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & \frac{12}{7} \end{pmatrix}$$
9. DETERMINANTS

\[ = 21 \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & \frac{-2}{7} \end{pmatrix} = (-\frac{2}{7})(21) \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} = -6. \]

**Remark.** When finding determinants, one usually row reduces until one reaches upper triangular form. The determinant of a matrix in upper triangular form is the product of the diagonal elements.

The determinant has several standard uses:

\begin{itemize}
  \item finding the volume of an \( n \)-dimensional box,
  \item finding cross products,
  \item testing for invertibility,
  \item finding eigenvalues.
\end{itemize}

**Remark.** Eigenvalues are fundamental to science and hence many methods exist for finding them numerically. Using the determinant to find eigenvalues is generally impractical. See [St] for information comparing methods and illustrating their advantages and disadvantages. At this stage of the book we are more concerned with theoretical issues than with computational methods. Row operations generally provide the best way to compute determinants, but determinants do not provide the best way to compute eigenvalues.

The following theorem summarizes the basic method of computing determinants. The function \( D \) is called the determinant function. The multi-linearity property includes both (13.2) and (13.3). The alternating property implies (13.1).

**Theorem 9.2.** Let \( M(n) \) denote the space of \( n \)-by-\( n \) matrices with elements in a field \( F \). There is a unique function \( D : M(n) \to F \), called the determinant, with the following properties:

\begin{itemize}
  \item \( D \) is linear in each row. (multi-linearity)
  \item \( D(I) = 1 \).
  \item \( D(A) = 0 \) if two rows of \( A \) are the same. (alternating property)
\end{itemize}

See [HK] (or any other good book on linear algebra) for a proof of this theorem. The idea of the proof is simple; the three properties force the determinant to satisfy the Laplace expansion from Theorem 9.3.

Over the real or complex numbers, for example, the third property from Theorem 9.2 can be restated as \( D(A') = -D(A) \) if \( A' \) is obtained from \( A \) by interchanging two rows. For general fields, however, the two properties are not identical!

**Example 9.2.** Suppose the field of scalars consists only of the two elements 0 and 1. Hence \( 1 + 1 = 0 \). The determinant of the two-by-two identity matrix is 1. If we interchange the two rows, then the determinant is still 1, since \(-1 = 1\) in this field. This example suggests why we state the alternating property as above.

The next result (Theorem 9.3) is of theoretical importance, but it should not be used for computing determinants. Recall that a permutation on \( n \)-letters is a bijection of the set \( \{1, \ldots, n\} \). The **signum** of a permutation \( \sigma \), written \( \text{sgn}(\sigma) \), is a measure of orientation: \( \text{sgn}(\sigma) = 1 \) if it takes an even number of transpositions to obtain the identity permutation, and \( \text{sgn}(\sigma) = -1 \) if it takes an odd number to do so.

Most books on linear or abstract algebra discuss why this number is well-defined. In other words, given a permutation, no matter what sequence of transpositions we use to convert it to the identity, the parity will be the same. Perhaps the easiest approach to this conclusion is to define \( \text{sgn} \) as follows:

\[
\text{sgn}(\sigma) = \frac{\prod_{1 \leq i < j \leq n} \sigma(i) - \sigma(j)}{\prod_{1 \leq i < j \leq n} (i - j)}.
\]  

(14)

Formula (14) implies that \( \text{sgn}(\sigma \circ \eta) = \text{sgn}(\sigma) \text{sgn}(\eta) \), from which the well-defined property follows. See Exercise 9.10.
Theorem 9.3 (Laplace expansion). Let $A$ be an $n$-by-$n$ matrix. Then
\[
\det(a_{ij}) = \sum_{\sigma} \text{sgn}(\sigma) \prod_{j=1}^{n} a_{j\sigma(j)},
\]
The sum is taken over all permutations $\sigma$ of $\{1, ..., n\}$.

The next theorem is of major theoretical importance. It is also important in various applications, such as the $ABCD$ matrices used in electrical engineering for transmission lines and the ray transfer matrices used in optical design. See Section 7 of Chapter 2.

Theorem 9.4. Suppose $A, B$ are square matrices of the same size. Then
\[
\det(AB) = \det(A) \det(B).
\]

One particular proof of this result is quite interesting. We indicate this proof, without filling in details. First one proves that $\det(B) = 0$ if and only if $B$ is not invertible. If $\det(B) = 0$, then $B$ is not invertible, and hence (for every $A$) $AB$ is not invertible. Therefore $\det(AB) = 0$ for all $A$, and the result holds. If $B$ is invertible, then $\det(B) \neq 0$, and we can consider the function taking the rows of $A$ to
\[
\frac{\det(AB)}{\det(B)}.
\]
One then shows that this function satisfies the properties of the determinant function, and hence (by the previous theorem) must be $\det(A)$.

Theorem 9.4 is of fundamental importance, for several reasons. A matrix is invertible if and only if it is a product or elementary row matrices. This statement allows us to break down the effect of an arbitrary invertible linear map into simple steps. The effect of these steps on, for example, volumes is easy to understand. Using Theorem 9.4 we can then prove that the determinant of an $n$-by-$n$ matrix is the oriented $n$-dimensional volume of the box spanned by the row or column vectors. In the next section we will discuss similarity, and we will use Theorem 9.4 to see that similar matrices have the same determinant.

Given a linear map on $\mathbb{R}^n$ or $\mathbb{C}^n$, we define its determinant to be the determinant of its matrix representation with respect to the standard basis. For a general finite-dimensional vector space $V$ and linear map $L : V \to V$, the determinant depends on the choice of bases. Whether the determinant is 0, however, is independent of the basis chosen. We can now summarize much of linear algebra in the following result. We have already established most of the equivalences, and hence we relegate the proof to the exercises.

Theorem 9.5. Let $V$ be a finite-dimensional vector space. Assume $L : V \to V$ is linear. Then the following statements hold or fail simultaneously:

1. $L$ is invertible.
2. $L$ is row equivalent to the identity.
3. There are elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k E_{k-1} \cdots E_1 L = I$.
4. There are elementary matrices $F_1, F_2, \ldots, F_k$ such that $L = F_1 F_2 \cdots F_k$.
5. $\mathcal{N}(L) = \{0\}$.
6. $L$ is injective.
7. $\mathcal{R}(L) = V$.
8. For any basis, the columns of the matrix of $L$ are linearly independent.
9. For any basis, the rows of the matrix of $L$ are linearly independent.
10. For any basis, the columns of the matrix of $L$ span $V$.
11. For any basis, the rows of the matrix of $L$ span $V$.
12. For any basis, $\det(L) \neq 0$.

Exercise 9.1. Show that the determinant of an elementary matrix is non-zero.
Exercise 9.2. Prove that items (5) and (6) in Theorem 9.5 hold simultaneously for any linear map $L$, even in infinite dimensions.

Exercise 9.3. Derive the equivalence of (5) and (7) from Theorem 6.1.

Exercise 9.4. Give an example (it must be infinite-dimensional) where item (5) holds in Theorem 9.5, but items (1) and (7) fail.

Exercise 9.5. Use row operations and Exercise 9.1 to prove that items (1) through (7) from Theorem 9.5 are equivalent.

Exercise 9.6. It is obvious that (8) and (10) from Theorem 9.5 are equivalent, and that (9) and (11) are equivalent. Why are all four statements equivalent?

Exercise 9.7. Graph the parallelogram with vertices at $(0,0)$, $(1,10)$, $(5,14)$, and $(6,24)$. Find its area by finding the determinant of
\[
\begin{pmatrix}
-1 & 10 \\
-5 & 14 \\
\end{pmatrix}
\]

Exercise 9.8. Use row operations to find the determinant of $A$:
\[
A = \begin{pmatrix}
1 & 2 & 3 & 4 \\
4 & 3 & 2 & 1 \\
2 & 4 & 7 & 8 \\
3 & 6 & 9 & 9 \\
\end{pmatrix}
\]

Exercise 9.9. Given $n$ distinct numbers $x_j$, form the matrix with entries $x_j^k$ for $0 \leq j, k \leq n-1$. Find its determinant. This matrix is called a Vandermonde matrix. Suggestion: Regard the matrix as a polynomial in the entries and determine where this polynomial must vanish.

Exercise 9.10. Use (14) to show that $\text{sgn}(\sigma \circ \eta) = \text{sgn}(\sigma) \text{sgn}(\eta)$.

Exercise 9.11. Consider the permutation $\sigma$ that reverses order. Thus $\sigma(j) = n+1-j$ for $1 \leq j \leq n$. Find $\text{sgn}(\sigma)$.

10. Diagonalization and generalized eigenspaces

A linear transformation $L$ is called diagonalizable if there is basis consisting of eigenvectors. In this case, the matrix of $L$ with respect to this basis is diagonal, with the eigenvalues on the diagonal. The matrix of $L$ with respect to an arbitrary basis will not generally be diagonal. If a matrix $A$ is diagonalizable, however, then there is a diagonal matrix $D$ and an invertible matrix $P$ such that
\[
A = PDP^{-1}. \tag{15}
\]
The columns of $P$ are the eigenvectors. See Example 5.1 for an illustration. More generally, matrices $A$ and $B$ are called similar if there is an invertible $P$ for which $A = PBP^{-1}$. Similarity is an equivalence relation. Note that $A$ is diagonalizable if and only if $A$ is similar to a diagonal matrix.

Formula (15) has many consequences. One application is to define and easily compute functions of $A$. For example, if $p(x) = \sum_j c_j x^j$ is a polynomial in one variable $x$, we naturally define $p(A)$ by
\[
p(A) = \sum_j c_j A^j.
\]
The following simple result enables us to compute polynomial functions of matrices.
Lemma 10.1. Let \( f(x) \) be a polynomial in one real (or complex) variable \( x \). Suppose \( A \) and \( B \) are similar matrices with \( A = PBP^{-1} \). Then \( f(A) \) and \( f(B) \) are similar, with
\[
f(A) = Pf(B)P^{-1}.
\] (16)

Proof. The key step in the proof is that \( (PBP^{-1})^j = PB^j P^{-1} \). This step is easily verified by induction; the case when \( j = 1 \) is trivial, and passing from \( j \) to \( j + 1 \) is simple:
\[
(PBP^{-1})^{j+1} = (PBP^{-1})^j PBP^{-1} = PB^j P^{-1} PBP^{-1} PBP^{-1} = PB^{j+1} P^{-1}.
\] (17)
The second step uses the induction hypothesis, and we have used associativity of matrix multiplication.

Formula (16) easily follows:
\[
f(A) = \sum c_j A^j = \sum c_j PB^j P^{-1} = P(\sum c_j B^j)P^{-1} = Pf(B)P^{-1}.
\]

Remark. Once we know (16) for polynomial functions, we can conclude an analogous result when \( f \) is a convergent power series. A proof of the following Corollary requires complex variable theory.

Corollary 10.2. Suppose the power series \( f(z) = \sum a_n z^n \) converges for \( |z| < R \). Let \( A : \mathbb{C}^n \to \mathbb{C}^n \) be a linear map whose eigenvalues \( \lambda \) all satisfy \( |\lambda| < R \). Then the series \( f(A) = \sum c_n A^n \) converges and defines a linear map. Furthermore, if \( A = PBP^{-1} \), then \( f(A) = Pf(B)P^{-1} \).

Two special cases are particularly useful. The series \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \) converges for all complex numbers \( z \) and defines the complex exponential function. Hence, if \( A \) is a square matrix, we define \( e^A \) by
\[
ee^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}.
\] (18)
The second example comes from the geometric series. If all the eigenvalues of \( A \) have magnitude smaller than 1, then \( I - A \) is invertible, and
\[
(I - A)^{-1} = \sum_{n=0}^{\infty} A^n.
\]
See Definition 5.8 of Chapter 4 for an alternative approach, using the Cauchy integral formula, to define functions of operators (linear transformations).

The next remark gives one of the fundamental applications of finding functions of operators. All constant coefficient homogeneous ODE reduce to exponentiating matrices.

Remark. Consider an \( n \)-th-order constant coefficient ODE for a function \( y \):
\[
y^{(n)} = c_{n-1} y^{(n-1)} + \cdots + c_1 y' + c_0 y.
\]
We can rewrite this equation as a first-order system, by giving new names to the derivatives, to obtain
\[
Y' = \begin{pmatrix} y' \\ y'' \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots \\ c_0 & c_1 & c_2 & \cdots & c_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix} = AY.
\]
The solution to the system is given by \( Y(t) = e^{At}Y(0) \). See Exercises 10.4 through 10.7 for examples.
How can we exponentiate a matrix? If $A$ is diagonal, with diagonal elements $\lambda_k$, then $e^A$ is diagonal with diagonal elements $e^{\lambda_k}$. If $A$ is diagonalizable, that is, similar to a diagonal matrix, then finding $e^A$ is also easy. If $A = PD P^{-1}$, then $e^A = Pe^D P^{-1}$. Not all matrices are diagonalizable. The simplest example is the matrix $A$, given by

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. \tag{19}$$

Exercise 10.1 shows how to exponentiate this particular $A$.

The matrix in (19) satisfies $A(e_1) = \lambda e_1$ and $A(e_2) = \lambda e_2 + e_1$. Thus $(A - \lambda I)(e_2) = e_1$ and hence $(A - \lambda I)^2(e_2) = (A - \lambda I)(e_1) = 0$.

The vector $e_2$ is an example of a generalized eigenvector. We have the following definition.

**Definition 10.3.** Let $L : V \to V$ be a linear map. The eigenspace $E_\lambda$ is the subspace defined by $E_\lambda = \{v : L v = \lambda v\} = \mathcal{N}(L - \lambda I)$.

The generalized eigenspace $K_\lambda$ is the subspace defined by $K_\lambda = \{v : (L - \lambda I)^k v = 0 \text{ for some } k\}$.

**Remark.** Both $E_\lambda$ and $K_\lambda$ are subspaces of $V$. Furthermore, $L(E_\lambda) \subseteq L(E_\lambda)$ and $L(K_\lambda) \subseteq K_\lambda$.

The following result clarifies both the notion of generalized eigenspace and what happens when the characteristic polynomial of a constant coefficient linear ODE has multiple roots.

**Theorem 10.4.** Let $L : V \to V$ be a linear map. Assume $(L - \lambda I)^{k-1}(v) \neq 0$, but $(L - \lambda I)^k v = 0$. Then, for $0 \leq j \leq k - 1$, the vectors $(L - \lambda I)^j(v)$ are linearly independent.

**Proof.** As usual, to prove that vectors are linearly independent, we assume that (the zero vector) 0 is a linear combination of them, and we show that all the coefficient scalars must be 0. Therefore assume

$$0 = \sum_{j=0}^{k-1} c_j(L - \lambda I)^j(v). \tag{20}$$

Apply $(L - \lambda I)^{k-1}$ to both sides of (20). The only term not automatically 0 is the term $c_0(L - \lambda I)^{k-1}v$. Hence this term is 0 as well; since $(L - \lambda I)^{k-1}(v) \neq 0$, we must have $c_0 = 0$. Now we can apply $(L - \lambda I)^{k-2}$ to both sides of (20) and obtain $c_1 = 0$. Continue in this fashion to obtain $c_j = 0$ for all $j$. \qed

Consider the subspace spanned by the vectors $(L - \lambda I)^jv$ for $0 \leq j \leq k - 1$. For simplicity we first consider the case $k = 2$. Suppose $A(e_1) = \lambda e_1$ and $A(e_2) = \lambda e_2 + e_1$. Then $(A - \lambda I)e_1 = 0$ whereas $(A - \lambda I)(e_2) \neq 0$ but $(A - \lambda I)^2(e_2) = 0$. Thus $e_1$ is an eigenvector and $e_2$ is a generalized eigenvector. Using this basis, we obtain the matrix

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}. $$

To make the pattern evident, we also write down the analogous matrix when $k = 5$.

Using the generalized eigenvectors as a basis, again in the reverse order, we can write $L$ as the matrix

$$L = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}. \tag{21}$$
Such a matrix is called a **Jordan block**; it cannot be diagonalized. Jordan blocks arise whenever a matrix has (over the complex numbers) too few eigenvectors to span the space. In particular, they arise in the basic theory of linear differential equations. We illustrate with the differential equation

\[(D - \lambda I)^2 y = y'' - 2\lambda y' + \lambda^2 y = 0.\]

We know that \(y(x) = e^{\lambda x}\) is one solution. We obtain the linearly independent solution \(xe^{\lambda x}\) by solving the equation \((D - \lambda I)y = e^{\lambda x}\). Thus \(e^{\lambda x}\) is an eigenvector of \(D\), and \(xe^{\lambda x}\) is a generalized eigenvector. More generally, consider the operator \((D - \lambda I)^k\). The solution space to \((D - \lambda I)^k y = 0\) is \(k\)-dimensional and is spanned by \(x^j e^{\lambda x}\) for \(0 \leq j \leq k - 1\). This well-known fact from differential equations illustrates the notion of generalized eigenvector. The independence of the functions \(x^j e^{\lambda x}\) follows from Theorem 8.4, but can also be checked directly. See Exercise 10.3.

To further develop the connection between linear algebra and constant coefficient differential equations, we will need to discuss more linear algebra.

**Exercise 10.1.** Find \(e^A\) where \(A\) is as in (19), by explicitly summing the series. Suggestion: Things work out easily if one writes \(A = \lambda I + N\) and notes that \(N^2 = 0\).

**Exercise 10.2.** Verify that \(E_\lambda\) and \(K_\lambda\), as defined in Definition 8.3, are invariant subspaces under \(L\). A subspace \(U\) is invariant under \(L\) if \(L(U) \subseteq U\).

**Exercise 10.3.** Prove that the functions \(x^j e^{\lambda x}\) for \(0 \leq j \leq n\) are linearly independent.

**Exercise 10.4.** Consider the ODE \(y''(t) + 4y(t) = 0\). By letting \(x(t) = y'(t)\), rewrite the ODE as a coupled system of first-order ODE in the form

\[
\begin{pmatrix}
x'
\end{pmatrix}
= A
\begin{pmatrix}
x
\end{pmatrix}.
\]

**Exercise 10.5.** Use linear algebra to find all solutions of the coupled system of ODE given by

\[
\begin{pmatrix}
x'
y'
\end{pmatrix}
= \begin{pmatrix}
0 & 4 \\
-1 & 4
\end{pmatrix}
\begin{pmatrix}
x
y
\end{pmatrix}.
\]

Also solve it by eliminating one of the variables and obtaining a second-order ODE for the other.

**Exercise 10.6.** Consider the coupled system of ODE given by \(x' = -y\) and \(y' = x\). The general solution is given by

\[
x(t) = a\cos(t) + b\sin(t),
\]

\[
y(t) = a\sin(t) - b\cos(t).
\]

First, derive this fact by eliminating \(x\) and \(y\). In other words, write the first-order system as a second-order equation and solve it. Next, derive this fact by writing

\[
\begin{pmatrix}
x'
y'
\end{pmatrix}
= \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
x
y
\end{pmatrix}
\]

and then exponentiating the matrix \(\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}\).

**Exercise 10.7.** Solve the previous exercise using complex variables, by putting \(z = x + iy\) and noting that the equation becomes \(z' = iz\). Comment: This exercise suggests the connection between the exponential function and trig functions elaborated in Chapter 2.
11. Characteristic and minimal polynomials

This section assumes some familiarity with complex numbers. We noted in Example 7.4 that a real matrix need not have real eigenvalues. We will therefore work with \( n \times n \) matrices with complex entries; such matrices always have complex eigenvalues. Some readers may wish to read Section 1 from Chapter 2 before reading this section.

**Definition 11.1.** Suppose \( L : \mathbb{C}^n \to \mathbb{C}^n \) is linear. Its **characteristic polynomial** \( p_L(z) \) is defined by

\[
p_L(z) = \det(L - zI).
\]

The characteristic polynomial is intrinsically associated with a linear map. Let \( A \) and \( B \) be square matrices. If \( B \) is similar to \( A \), then \( A \) and \( B \) have the same characteristic polynomial. See Lemma 9.2.

We have seen that the roots of \( p_A \) are the eigenvalues of \( A \). Hence the set of eigenvalues of \( B \), including multiplicities, is the same as that of \( A \) when \( B \) is similar to \( A \). More generally, any symmetric function of the eigenvalues is unchanged under a similarity transformation \( A \mapsto PAP^{-1} \).

We benefit from working over \( \mathbb{C} \) here because each polynomial \( p(z) \) with complex coefficients splits into linear factors. The proof of this statement, called the **fundamental theorem of algebra** for historical reasons, requires complex analysis, and we postpone it until Chapter 4. If we work over the real numbers, then even simple examples such as \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) have no real eigenvalues. The matrix \( J \) rotates a vector by ninety degrees, and hence has no eigenvalues. See the discussion in Example 7.4 and at the end of the first section of Chapter 2.

Since each polynomial factors over \( \mathbb{C} \), we can write the characteristic polynomial of \( A \) as

\[
p_A(z) = \pm \prod_j (z - \lambda_j)^{m_j}.
\]

The exponent \( m_j \) is called the **multiplicity** of the eigenvalue \( \lambda_j \).

The famous Cayley-Hamilton theorem states for each \( L \) that \( p_L(L) = 0 \); in other words, \( p_L \) annihilates \( L \). The characteristic polynomial need not be the polynomial of smallest degree that annihilates \( L \). The **minimal polynomial** of \( L \) is the unique monic polynomial \( m_L \) of smallest degree for which \( m_L(L) = 0 \). It follows that \( m_L \) divides \( p_L \). In abstract linear algebra, the definition of \( m_L \) is sometimes stated a bit differently, but the meaning is the same. We give several simple examples.

**Example 11.1.** Put \( L = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} \). Then \( p_L(z) = (z - 3)^2 \) and \( m_L(z) = (z - 3)^2 \).

Put \( L = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \). Then \( p_L(z) = (z - 3)^2 \) but \( m_L(z) = (z - 3) \).

**Example 11.2.** Put

\[
L = \begin{pmatrix}
4 & 1 & 0 & 0 & 0 & 0 \\
0 & 4 & 1 & 0 & 0 & 0 \\
0 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 0 & 9 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}.
\]

Then \( p_L(z) = (z - 4)^3 (z - 9)(z - 3)^2 \) whereas \( m_L(z) = (z - 4)^2 (z - 9)(z - 3) \).

The following simple result about similarity shows that the characteristic and minimal polynomials are properties of the equivalence class under similarity.
Lemma 11.2. Assume $A$ and $B$ are square matrices with $B = P A P^{-1}$. Then $p_A = p_B$ and $m_A = m_B$.

Proof. Assume $B = P A P^{-1}$. Then:
\[
\det(B - \lambda I) = \det(P(A - \lambda I)P^{-1}) = \det(P)\det(A - \lambda I)\det(P^{-1}) = \det(A - \lambda I).
\]
Hence $p_B = p_A$. Suppose next that $q$ annihilates $A$. Then
\[
q(B) = \sum c_k B^k = \sum c_k P A^k P^{-1} = P(\sum c_k A^k) P^{-1} = P q(A) P^{-1} = 0.
\]
Therefore $q$ annihilates $B$ also. We could have also applied Lemma 8.1. By the same argument, if $q$ annihilates $B$, then $q$ annihilates $A$. Hence the set of polynomials annihilating $B$ is the same as the set of polynomials annihilating $A$, and therefore $m_A = m_B$. \[\square\]

Definition 11.3. The trace of an $n$-by-$n$ matrix is the sum of its diagonal elements.

The following important result is quite simple to prove; see Exercise 11.4.

Theorem 11.4. For square matrices $A, B$ of the same size, $\text{trace}(AB) = \text{trace}(BA)$. Furthermore, if $A$ and $C$ are similar, then $\text{trace}(A) = \text{trace}(C)$.

Corollary 11.5. If $A$ is an $n$-by-$n$ matrix of complex numbers, then $\text{trace}(A)$ is the sum of its eigenvalues.

The following theorem is a bit harder to prove. See Exercise 11.5.

Theorem 11.6. Assume that $L : \mathbb{C}^n \to \mathbb{C}^n$ is linear. Suppose that its distinct eigenvalues are $\lambda_1, ..., \lambda_k$. Then $L$ is diagonalizable if and only if
\[
m_L(z) = (z - \lambda_1)(z - \lambda_2)...(z - \lambda_k).
\]

For $L$ to be diagonalizable, the multiplicities of the eigenvalues in its characteristic polynomial can exceed 1, but the multiplicities in the minimal polynomial must each equal 1.

Exercise 11.1. Give an example of a matrix whose minimal polynomial is $(z - 4)^3(z - 9)^2(z - 3)$ and whose characteristic polynomial is $(z - 4)^3(z - 9)^2(z - 3)^3$.

Exercise 11.2. Find the characteristic polynomial of the three-by-three matrix
\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}.
\]

(Harder) Find the characteristic polynomial of the $n$-by-$n$ matrix
\[
\begin{pmatrix}
1 & 2 & ... & n \\
& n+1 & n+2 & ... & 2n \\
& & & & \\
n^2 - n + 1 & n^2 - n + 2 & ... & n^2
\end{pmatrix}.
\]

Suggestion: First note that 0 is an eigenvalue of multiplicity $n - 2$. Then find the eigenvectors corresponding to the non-zero eigenvectors. You might also need to evaluate $\sum_{j=1}^n j^2$.

Exercise 11.3. Find the determinant of an $n$-by-$n$ matrix whose diagonal entries all equal $x$ and whose other entries all equal 1. Suggestion: consider a matrix all of whose entries are 1.

Exercise 11.4. Prove Theorem 11.4. The first statement is a simple computation using the definition of matrix multiplication. The second statement follows from the first and the definition of similarity.
**Exercise 11.5.** Prove Theorem 11.6.

**Exercise 11.6.** Let the characteristic polynomial for some \( L \) be

\[
p_L(z) = \sum_{j=0}^{n} c_j z^j = (-1)^n \prod_{k=1}^{n} (z - \lambda_k).
\]

What are \( c_0 \) and \( c_{n-1} \)? Can you explain what \( c_j \) is for other values of \( j \)?

### 12. Similarity

Consider the equation \( A = PDP^{-1} \) defining similarity. This equation enables us to express polynomial functions (or convergent power series) of \( A \) by \( f(A) = P f(D) P^{-1} \). When \( D \) is easier to understand than \( A \), computations using \( D \) become easier than those using \( A \). The author calls this idea *Einstein’s principle*; when doing any problem, choose coordinates in which computation is easiest. We illustrate this idea with a problem from high school math.

Suppose \( f(x) = mx + b \) is an affine function on the real line. What happens if we iterate \( f \) many times? We write \( f^{(n)}(x) \) for the \( n \)-th iterate of \( f \) at \( x \). The reader can easily check that

\[
\begin{align*}
 f^{(2)}(x) &= f(f(x)) = m(f(x) + b) + b = m(mx + b) + b = m^2 x + (m + 1) b; \\
 f^{(3)}(x) &= f(f^2(x)) = m(m^2 x + (m + 1) b) + b = m^3 + (m^2 + m + 1) b.
\end{align*}
\]

From these formulas one can guess the general situation and prove it by induction.

There is a much better approach to this problem. We try to write \( f(x) = mx + b \) so \( b = c(1 - m) \). If \( m = 1 \), then \( f(x) = x + b \); thus \( f \) is a translation by \( b \). Iterating \( n \) times results in translating by \( nb \). Hence \( f^{(n)}(x) = x + nb \) in this case.

If \( m \neq 1 \), then we can solve \( c - mc = b \) for \( c \) and write \( f(x) = mx - c + c \). Then \( f \) translates by \( -c \), multiplies by \( m \), and then translates back by \( c \). Thus \( f = TMT^{-1} \), where \( T \) translates and \( M \) multiplies. We immediately obtain

\[
f^{(n)}(x) = TM^n T^{-1}(x) = m^n (x - c) + c = m^n x + b \frac{1 - m^n}{1 - m}.
\]

Notice that we can recover the case \( m = 1 \) by letting \( m \) tend to \( 1 \) in (22).

Figure 8 illustrates the idea. Iterating \( f \) amounts to going across the top line. We can also go down, multiply, go up, go back down, multiply, etc. Going up and back down cancel out. Hence \( f^{(n)} = TM^n T^{-1} \). Iterating \( M \) amounts to going across the bottom line.

![Figure 8. Iteration](image-url)

We summarize the idea. Whenever an operation can be expressed as \( TMT^{-1} \), iterating the operation is easy. We need only iterate \( M \). More generally we can consider functions of the operation, by applying those functions to the operator \( M \). This idea explains the usefulness of many mathematical concepts in engineering, including diagonalizing matrices, the Fourier and Laplace transforms, and the use of linear fractional transformations in analyzing networks.

In the context of linear algebra, certain objects (characteristic polynomial, eigenvalues, trace, determinant, etc.) are invariant under similarity. These objects are *intrinsically* associated with the given linear map.
and hence are likely to have physical meaning. Other expressions (the sum of all the entries of a matrix, how many entries are 0, etc.) are artifacts of the coordinates chosen. They distract the observer from the truth.

**Remark.** Consider a linear transformation between spaces of different dimensions. It does not then make sense to consider eigenvectors and eigenvalues. Nonetheless, these notions are so crucial that we need to find their analogues in this more general situation. Doing so leads to the notions of singular values and the **singular value decomposition**. Although we could discuss this situation now, we postpone this material until Section 5 of Chapter 6, after we have defined adjoints.

**Exercise 12.1.** Suppose \( f(x) = Ax + b \) for \( x \in \mathbb{C}^n \), where \( A : \mathbb{C}^n \to \mathbb{C}^n \) is linear. Under what condition does the analogue of (22) hold?

### 13. Additional remarks on ODE

We saw in Section 8 how to solve a constant coefficient linear ordinary differential equation, by exponentiating a matrix. In this section we revisit such equations and provide an alternative manner for solving them, in order to apply the ideas on similarity.

Consider the general such equation
\[
y^{(n)} + c_{n-1}y^{(n-1)} + \cdots + c_1 y' + c_0 y = f. \tag{23}
\]
Here we allow the coefficients \( c_j \) to be real or complex numbers. We rewrite (23) in the form
\[
p(D)y = f,
\]
where \( p \) is the polynomial
\[
p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_0,
\]
and \( p(D) \) means that we substitute the operator \( D \) of differentiation for \( z \). In Chapter 4 we prove the fundamental theorem of algebra, which implies that we can factor a polynomial into linear factors:
\[
p(z) = \prod_{j=1}^{n} (z - \lambda_j). \tag{24}
\]
Here the \( \lambda_j \) are complex numbers, which can be repeated. Hence \( p(D) = \prod_{j=1}^{n} (D - \lambda_j) \). Hence to solve \( p(D)y = f \), it suffices to solve \( (D - \lambda)g = h \) and then iterate. In other words
\[
p(D)^{-1} = \prod (D - \lambda_j)^{-1}. \tag{25}
\]

In (25), the order of the factors does not matter, because
\[
(D - \lambda_1)(D - \lambda_2) = (D - \lambda_2)(D - \lambda_1).
\]

**Proposition 13.1.** Let \( D - \lambda \) be a first-order ordinary differential operator. Let \( M_\lambda \) denote the operation of multiplication by \( e^{\lambda t} \). Let \( J \) denote integration. Then, \( (D - \lambda)^{-1} = M_\lambda J M_\lambda^{-1} \). In more concrete language, we solve \( g' - \lambda g = h \) by writing
\[
g(t) = e^{\lambda t} \int_{t_0}^{t} e^{-\lambda u} h(u) du. \tag{26}
\]

**Proof.** One discovers formula (26) by variation of parameters. To prove the proposition, it suffices to check that \( g' - \lambda g = h \), which is a simple calculus exercise. \( \square \)

**Corollary 13.2.** We solve \( p(D)y = f \) by factoring \( p \) and iterating Proposition 13.1.

**Exercise 13.1.** Verify that (26) solves \( g' - \lambda g = h \).
**Exercise 13.2.** Discover (26) as follows. Assume that \( g(t) = c(t)e^{\lambda t} \), plug in the equation, and determine \( c \).

**Exercise 13.3.** Solve \((D - \lambda_1)(D - \lambda_2)y = f\) using Corollary 13.2.

**Exercise 13.4.** Use Proposition 13.1 to easily solve \((D - \lambda)^ky = f\). In particular, revisit Example 4.3 using this approach.