Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 7.3, 7.5, 7.6, 9.1, 9.2 in *The Way of Analysis*, Chapter 1 in *Calculus on Manifolds*.

You should do the following problems, but you do not need to hand in your solutions:

1. Suppose that \(<f_n>\) is a uniformly convergent sequence of functions with limit \(f\).
   Show that, if the \(\{f_n\}\) have, at worst, jump discontinuities, then so does the limit function.

2. Step functions are defined in problem 2. Prove the following facts about step functions:
   (a) A step function is Riemann integrable.
   (b) There is a representation of a step function so that the intervals \([a_j, b_j]\) are disjoint.

3. Let \(x_1, x_2 \in \mathbb{R}\) and let \((a_0, \ldots, a_k), (b_0, \ldots, b_k)\) be \((k+1)\)-tuples of real numbers.
   Show that there is a unique polynomial of degree at most \(2k + 1\) such that
   \[p^{[j]}(x_1) = a_j \quad \text{and} \quad p^{[j]}(x_2) = b_j \quad \text{for} \quad j = 0, \ldots, k.\]
   
   Hint: Consider the polynomials \((x - x_1)^k (x - x_2)^{k+1}, (x - x_1)^{k+1} (x - x_2)^k\) and argue by induction.

The following problems should be carefully written up and handed in.

1. The support of a function \(f\), is defined to be
   \[\text{supp } f = \{x : f(x) \neq 0\}.\]
   This is the smallest closed set containing the set \(\{x : f(x) \neq 0\}\).
   (a) Prove that a continuous function has compact support if and only if it vanishes outside of a bounded interval.
   (b) Show that if \(f\) and \(g\) are continuous, then
   \[\text{supp } f * g \subset \text{supp } f + \text{supp } g,\]
   where, for sets \(A, B \subset \mathbb{R}\), we define \(A + B = \{x + y : x \in A, y \in B\}\).

2. Recall that if \(E\) is a set, then the characteristic function of \(E\) is defined to be
   \[\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E. \end{cases}\]
A function, $g$, is called a step function if, for some $N \in \mathbb{N}$,

$$g(x) = \sum_{j=1}^{N} c_j \chi_{[a_j, b_j)}(x),$$

for real numbers $\{a_j < b_j : j = 1 \ldots, N\} \cup \{c_j : j = 1 \ldots, N\}$.

(a) Let $f$ be a Riemann integrable function. Given $\epsilon > 0$ show that there is a step function $g$ so that

$$\int |f(x) - g(x)| \, dx < \epsilon$$

(b) Use the previous result to show that if $f$ is a Riemann integrable function, and $\epsilon > 0$, then there is a continuous function $h$ so that

$$\int |f(x) - h(x)| \, dx < \epsilon$$

(c) Show that the supp $h$ can be taken to be an arbitrarily small neighborhood of the supp $f$.

3. In this problem we prove the following fact: If $f$ and $g$ are Riemann integrable, then $f \ast g$ is continuous.

(a) Show that if $< f_n >$ is a sequence of Riemann integrable functions and $f, g$ are other Riemann integrable functions, such that

$$\lim_{n \to \infty} \int |f_n(x) - f(x)| \, dx = 0,$$

then $< f_n \ast g >$ converges uniformly to $f \ast g$. You can assume that all of these functions vanish outside a fixed finite interval.

(b) Show that if $f$ is continuous and $g$ is Riemann integrable, then $f \ast g$ is continuous.

(c) Use 2b and 3a,b to prove the statement above: if $f$ and $g$ are Riemann integrable, then $f \ast g$ is a continuous function.

4. Let $\mathcal{P}_n$ denote polynomials of degree at most $n$. Let $\{x_1, \ldots, x_{n+1}\}$ be distinct points in $\mathbb{R}$. Show that the function

$$N(p) = \max\{|p(x_j)| : j = 1, \ldots, n + 1\}$$

defines a norm on $\mathcal{P}_n$. For any $x \in \mathbb{R}$, show that there is a constant $C_x$, so that

$$|p(x)| \leq C_x N(p) \text{ for all } p \in \mathcal{P}_n.$$

5. Let $\mathcal{M}_{m,n}$ denote the vector space of real $m \times n$ matrices. Show that the function

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$$

defines a norm on $\mathcal{M}_{m,n}$. Here $\| \cdot \|_2$ denotes the usual Euclidean norm on a vector space.
6. Let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be a nested sequence of non-empty compact subsets of $\mathbb{R}^n$ prove that

$$\bigcap_{j=1}^{\infty} A_j \neq \emptyset.$$ 

Give an example that shows that compactness is necessary.