Math 509  
Problem set 1, due January 23, 2017  
Dr. Epstein  

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.  
For this week, read Chapter 7.3, 7.5, 7.6, 9.1, 9.2 in *The Way of Analysis*. 

You should do the following problems, but you do not need to hand in your solutions: 

1. Suppose that $< f_n >$ is a uniformly convergent sequence of functions with limit $f$. Show that, if the $\{f_n\}$ have, at worst, jump discontinuities, then so does the limit function.

2. Step functions are defined in problem 2. Prove the following facts about step functions:
   (a) A step function is Riemann integrable.
   (b) There is a representation of a step function so that the intervals $\{[a_j, b_j]\}$ are disjoint.

The following problems should be carefully written up and handed in.  
If $A \subset \mathbb{R}$, then a neighborhood of $A$ is a open set $U \supset A$.

1. Let $x_1, x_2 \in \mathbb{R}$ and let $(a_0, \ldots, a_k), (b_0, \ldots, b_k)$ be $(k+1)$-tuples of real numbers. Show that there is a unique polynomial of degree at most $2k + 1$ such that 

   \[ p^{[j]}(x_1) = a_j \quad \text{and} \quad p^{[j]}(x_2) = b_j \quad \text{for} \quad j = 0, \ldots, k. \]

   Hint: Consider the polynomials $(x - x_1)^k(x - x_2)^{k+1}, (x - x_1)^{k+1}(x - x_2)^k$ and argue by induction.

2. The support of a function $f$, is defined to be 

   \[ \text{supp } f = \{x : f(x) \neq 0\}. \]

   That is, the smallest closed set containing the set $\{x : f(x) \neq 0\}$.

   (a) Prove that a continuous function has compact support if and only if it vanishes outside of a bounded interval.
   (b) Show that if $f$ and $g$ are continuous, then 

   \[ \text{supp } f * g \subset \text{supp } f + \text{supp } g, \]

   where, for sets $A, B \subset \mathbb{R}$, we define $A + B = \{x + y : x \in A, y \in B\}$.

3. Recall that if $E$ is a set, then the characteristic function of $E$ is defined to be 

   \[ \chi_E(x) = \begin{cases} 
   1 & \text{if } x \in E \\
   0 & \text{if } x \notin E.
   \end{cases} \]
A function, $g$, is called a step function if, for some $N \in \mathbb{N}$,

$$g(x) = \sum_{j=1}^{N} c_j \mathcal{X}_{[a_j, b_j]}(x),$$

for real numbers $\{a_j < b_j : j = 1 \ldots, N\} \cup \{c_j : j = 1 \ldots, N\}$.

(a) Let $f$ be a Riemann integrable function. Given $\epsilon > 0$ show that there is a step function $g$ so that

$$\int |f(x) - g(x)| \, dx < \epsilon$$

(b) Use the previous result to show that if $f$ is a Riemann integrable function, and $\epsilon > 0$, then there is a continuous function $h$ so that

$$\int |f(x) - h(x)| \, dx < \epsilon$$

(c) Show that the supp $h$ can be taken to be an arbitrarily small neighborhood of the supp $f$.

4. In this problem we prove the following fact: If $f$ and $g$ are Riemann integrable, then $f \ast g$ is continuous.

(a) Show that if $< f_n >$ is a sequence of Riemann integrable functions and $f, g$ are other Riemann integrable functions, such that

$$\lim_{n \to \infty} \int |f_n(x) - f(x)| \, dx = 0,$$

then $< f_n \ast g >$ converges uniformly to $f \ast g$. You can assume that all of these functions vanish outside a fixed finite interval.

(b) Show that if $f$ is continuous and $g$ is Riemann integrable, then $f \ast g$ is continuous.

(c) Use 2b and 3a,b to prove the statement above: if $f$ and $g$ are Riemann integrable $f \ast g$ is a continuous function.

5. Suppose that $\{f_n\}$ is a set of functions defined on an interval $I$, for which there exist positive constants $\alpha, M$ so that, for all $n$,

$$|f_n(x) - f_n(y)| \leq M |x - y|^{\alpha} \text{ for all } x, y \in I.$$  

Show that the family $\{f_n\}$ is uniformly equicontinuous.

6. Give an example of a sequence of uniformly bounded, uniformly equicontinuous functions defined on $\mathbb{R}$, which does not have any uniformly convergent sub-sequences.