You should do the following problems, but you do not need to hand in your solutions:

1. Suppose that \( < f_n > \) is a uniformly convergent sequence of functions with limit \( f \). Show that, if the \( \{ f_n \} \) have, at worst, jump discontinuities, then so does the limit function.

2. Step functions are defined in problem 2. Prove the following facts about step functions:
   (a) A step function is Riemann integrable.
   (b) There is a representation of a step function so that the intervals \( \{ [a_j, b_j] \} \) are disjoint.

3. Let \( x_1, x_2 \in \mathbb{R} \) and let \( (a_0, \ldots, a_k), (b_0, \ldots, b_k) \) be \( (k + 1) \)-tuples of real numbers. Show that there is a unique polynomial of degree at most \( 2k + 1 \) such that
   \[ p^{[j]}(x_1) = a_j \text{ and } p^{[j]}(x_2) = b_j \text{ for } j = 0, \ldots, k. \]
   Hint: Consider the polynomials \( (x - x_1)^k(x - x_2)^{k+1}, (x - x_1)^{k+1}(x - x_2)^k \) and argue by induction.

The following problems should be carefully written up and handed in.

1. The support of a function \( f \), is defined to be
   \[ \text{supp } f = \{ x : f(x) \neq 0 \}. \]
   This is the smallest closed set containing the set \( \{ x : f(x) \neq 0 \} \).
   (a) Prove that a continuous function has compact support if and only if it vanishes outside of a bounded interval.
   (b) Show that if \( f \) and \( g \) are continuous, and at least one has bounded support, then
   \[ \text{supp } f * g \subseteq \text{supp } f + \text{supp } g, \]
   where, for sets \( A, B \subseteq \mathbb{R} \), we define \( A + B = \{ x + y : x \in A, y \in B \} \).

2. Recall that if \( E \) is a set, then the characteristic function of \( E \) is defined to be
   \[ \chi_E(x) = \begin{cases} 
   1 & \text{if } x \in E \\
   0 & \text{if } x \notin E.
   \end{cases} \]
A function, \( g \), is called a step function if, for some \( N \in \mathbb{N} \),
\[
g(x) = \sum_{j=1}^{N} c_j \chi_{[a_j, b_j]}(x),
\]
for real numbers \( \{a_j < b_j : j = 1 \ldots, N\} \cup \{c_j : j = 1 \ldots, N\} \).

(a) Let \( f \) be a Riemann integrable function. Given \( \epsilon > 0 \) show that there is a step function \( g \) so that
\[
\int |f(x) - g(x)| \, dx < \epsilon
\]
(b) Use the previous result to show that if \( f \) is a Riemann integrable function, and \( \epsilon > 0 \), then there is a continuous function \( h \) so that
\[
\int |f(x) - h(x)| \, dx < \epsilon
\]
(c) Show that the \( \text{supp} \ h \) can be taken to lie in an arbitrarily small neighborhood of the \( \text{supp} \ f \). This means that, given \( \epsilon > 0 \), there exists a function \( h \) so that
\[
\text{supp} \ h \subset \bigcup_{x \in \text{supp} \ f} B_\epsilon(x).
\]

3. In this problem we prove the following fact: If \( f \) and \( g \) are Riemann integrable, then \( f \ast g \) is continuous. You can assume that all functions in this problem vanish outside a fixed finite interval.

(a) Show that if \( \langle f_n \rangle \) is a sequence of Riemann integrable functions and \( f, g \) are other Riemann integrable functions, such that
\[
\lim_{n \to \infty} \int |f_n(x) - f(x)| \, dx = 0,
\]
then \( \langle f_n \ast g \rangle \) converges uniformly to \( f \ast g \).

(b) Show that if \( f \) is continuous and \( g \) is Riemann integrable, then \( f \ast g \) is continuous.

(c) Use 2b and 3a,b to prove the statement above: if \( f \) and \( g \) are Riemann integrable, then \( f \ast g \) is a continuous function.

4. Let \( \mathcal{P}_n \) denote polynomials of degree at most \( n \). Let \( \{x_1, \ldots, x_{n+1}\} \) be distinct points in \( \mathbb{R} \). Show that the function
\[
N(p) = \max\{|p(x_j)| : j = 1, \ldots, n + 1\}
\]
defines a norm on \( \mathcal{P}_n \). For any \( x \in \mathbb{R} \), show that there is a constant \( C_x \), so that
\[
|p(x)| \leq C_x N(p) \text{ for all } p \in \mathcal{P}_n.
\]

5. Let \( \mathcal{M}_{m,n} \) denote the vector space of real \( m \times n \) matrices. Show that the function
\[
\|A\| = \sup_{x \in \mathbb{R}^n \neq 0} \frac{\|A \cdot x\|_2}{\|x\|_2}
\]
defines a norm on $\mathcal{M}_{m,n}$. Here $\| \cdot \|_2$ denotes the usual Euclidean norm on a vector space, and $A \cdot x$ is matrix-vector multiplication.

6. Let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be a nested sequence of non-empty compact subsets of $\mathbb{R}^n$ prove that

$$\bigcap_{j=1}^{\infty} A_j \neq \emptyset.$$ 

Give an example that shows that compactness is necessary.