Math 509
Problem set 3, due February 12, 2019
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 7.4, Chapter 8 and Chapter 9.1.3 in **The Way of Analysis**.

You should do the following problems, but you do not need to hand in your solutions:

1. Do the following complex arithmetic problems
   (a) $(1+3i)(4-2i) =?$
   (b) $1/(3-8i) =?$
   (c) Find all complex numbers z so that $z^2 = i$.
2. Prove that $e^{xy} = (e^x)^y$. Using this, for $x > 0$, define the function $x^x = e^{x\log x}$. Show that x^x is differentiable in $(0, \infty)$, and compute its derivative. What is $\lim_{x\to 0^+} x^x$?

The following problems should be carefully written up and handed in.

1. Define the map $T : \mathcal{C}^0([0,1]) \to \mathcal{C}^0([0,1])$ by
$$Tf(x) = x + \int_0^x tf(t)dt.$$
   (a) Show that there is a radius $0 < r$ so that $T : B_r(0) \to B_r(0)$, and that T is a contraction on this ball. We use the sup-norm on $\mathcal{C}^0([0,1])$.
   (b) Show that $T(B_r(0))$ is, in fact, a precompact subset of $\mathcal{C}^0([0,1])$, that is its closure is a compact set.
   (c) Explain why T has a fixed point in this ball, even though $\dim \mathcal{C}^0([0,1]) = \infty$. Prove this directly, do not simply quote the theorem in the book.
   (d) Show that a fixed point of T actually belongs to $\mathcal{C}^1([0,1])$.
   (e) Find the fixed point. Hint: Differentiate! The final answer is expressed as an indefinite integral, which cannot done explicitly.
2. Recall that the Hermitian inner product on $\mathbb{C}^n$ is defined by
$$\langle \boldsymbol{z}, \boldsymbol{w}\rangle = \sum_{j=1}^n z_j \bar{w}_j. \tag{1}$$
   Prove the polarization identity
$$\langle \boldsymbol{z}, \boldsymbol{w}\rangle = \frac{1}{4}\left[\|\boldsymbol{z}+\boldsymbol{w}\|^2 - \|\boldsymbol{z}-\boldsymbol{w}\|^2 + i\,\|\boldsymbol{z}+i\,\boldsymbol{w}\|^2 - i\,\|\boldsymbol{z}-i\,\boldsymbol{w}\|^2\right]. \tag{2}$$
3. For a a positive real number, prove that the power series
$$f(x) = 1 + \sum_{j=1}^{\infty} \frac{a\cdots(a+1-j)}{j!}x^j \tag{3}$$

converges in $(-1,1)$. Prove that it converges to the function $(1+x)^a$.

4. For each of the following power series describe the exact subset of $\mathbb{C}$ where it converges.
   (a) $\sum_{j=1}^{\infty} z^j$.
   (b) $\sum_{j=1}^{\infty} \frac{z^j}{j}$.
   (c) $\sum_{j=1}^{\infty} \frac{z^j}{j^2}$.
5. Suppose that $f(x)$ is a function represented by a convergent power series in the interval $(-1,1)$.
   (a) Prove: If there is a sequence of distinct points, $< x_j >$, in $(-1,1)$ such that $\lim_{j\to\infty} x_j = 0$, and $f(x_j) = 0$, then $f(x) = 0$ for all $x \in (-1,1)$. Hint: Show $f(0) = 0$, consider $f(x)/x$, and repeat.
   (b) Show that if $f^{[k]}(0) = 0$, for all non-negative integers k, then $f(x) = 0$ for all $x \in (-1,1)$.
6. Let

$$f(z) = \sum_{j=0}^{\infty} a_j z^j \tag{4}$$

be a convergent power series with real coefficients. Prove that

$$\overline{f(z)} = f(\bar{z}). \tag{5}$$

If (5) holds does it imply that the coefficient in the power series expansion are real?