Problem set 3, due February 12, 2019
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 7.4, Chapter 8 and Chapter 9.1.3 in The Way of Analysis.

You should do the following problems, but you do not need to hand in your solutions:

1. Do the following complex arithmetic problems
   (a) \((1 + 3i)(4 - 2i) = ?\)
   (b) \(1/(3 - 8i) = ?\)
   (c) Find all complex numbers \(z\) so that \(z^2 = i\).

2. Prove that \(e^{xy} = (e^x)^y\). Using this, for \(x > 0\), define the function \(x^x = e^{x \log x}\). 
   Show that \(x^x\) is differentiable in \((0, \infty)\), and compute its derivative. 
   What is \(\lim_{x \to 0^+} x^x\)?

The following problems should be carefully written up and handed in.

1. Define the map \(T: C^0([0, 1]) \to C^0([0, 1])\) by
   \[
   Tf(x) = x + \int_0^x tf(t)dt.
   \]
   (a) Show that there is a radius \(0 < r\) so that \(T: B_r(0) \to B_r(0)\), and that \(T\) is a contraction on this ball. 
   We use the sup-norm on \(C^0([0, 1])\).
   (b) Show that \(T(B_r(0))\) is, in fact, a precompact subset of \(C^0([0, 1])\), 
   that is its closure is a compact set.
   (c) Explain why \(T\) has a fixed point in this ball, even though \(\dim C^0([0, 1]) = \infty\).
   Prove this directly, do not simply quote the theorem in the book.
   (d) Show that a fixed point of \(T\) actually belongs to \(C^1([0, 1])\).
   (e) Find the fixed point. Hint: Differentiate! The final answer is expressed as an 
   indefinite integral, which cannot done explicitly.

2. Recall that the Hermitian inner product on \(\mathbb{C}^n\) is defined by
   \[
   \langle z, w \rangle = \sum_{j=1}^n \bar{z}_j \bar{w}_j.
   \]
   Prove the polarization identity
   \[
   \langle z, w \rangle = \frac{1}{4} \left[ \|z + w\|^2 - \|z - w\|^2 + i \|z + iw\|^2 - i \|z - iw\|^2 \right].
   \]

3. For a a positive real number, prove that the power series
   \[
   f(x) = 1 + \sum_{j=1}^{\infty} \frac{a \cdots (a + 1 - j)}{j!} x^j
   \]
converges in $(-1, 1)$. Prove that it converges to the function $(1 + x)^a$.

4. For each of the following power series describe the exact subset of $\mathbb{C}$ where it converges.
   (a) $\sum_{j=1}^{\infty} z^j$.
   (b) $\sum_{j=1}^{\infty} \frac{z^j}{j}$.
   (c) $\sum_{j=1}^{\infty} \frac{z^j}{j^2}$.

5. Suppose that $f(x)$ is a function represented by a convergent power series in the interval $(-1, 1)$.
   (a) Prove: If there is a sequence of distinct points, $\langle x_j \rangle$, in $(-1, 1)$ such that $\lim_{j \to \infty} x_j = 0$, and $f(x_j) = 0$, then $f(x) = 0$ for all $x \in (-1, 1)$. Hint: Show $f(0) = 0$, consider $f(x)/x$, and repeat.
   (b) Show that if $f^{[k]}(0) = 0$, for all non-negative integers $k$, then $f(x) = 0$ for all $x \in (-1, 1)$.

6. Let

   \begin{equation}
   f(z) = \sum_{j=0}^{\infty} a_j z^j
   \end{equation}

   be a convergent power series with real coefficients. Prove that

   \begin{equation}
   \overline{f(z)} = f(\overline{z}).
   \end{equation}

   If (5) holds does it imply that the coefficient in the power series expansion are real?