Math 509
Problem set 4, due February 19, 2019
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 12 in **The Way of Analysis**. The first two chapters of Stein and Shakarchi's book on Fourier Analysis are another good reference for this material.

You should do the following problems, but you do not need to hand in your solutions:

1. Suppose that f is a 2π-periodic, Riemann integrable function defined on $\mathbb{R}$. Show that the integral

$$\int_x^{x+2\pi} f(y)dy \tag{1}$$

does not depend on x.

2. If $< f_k >$ is a sequence of 2π-periodic, continuous funtions converging uniformly to a function f, then show that

$$\hat{f}(n) = \lim_{k\to\infty} \hat{f_k}(n) \text{ for all } n \in \mathbb{Z}. \tag{2}$$

The following problems should be carefully written up and handed in.

1. Let f be a 2π-periodic, Riemann integrable function.
   (a) Prove that if f is real valued, then

$$\hat{f}(n) = \overline{\hat{f}(-n)} \text{ for all } n \in \mathbb{Z}. \tag{3}$$

   (b) Prove that if f is even ($f(x) = f(-x)$) then $\hat{f}(n) = \hat{f}(-n)$, for all $n \in \mathbb{Z}$.
   (c) Show that if $f(x+\pi) = f(x)$, then $\hat{f}(2n+1) = 0$, for all $n \in \mathbb{Z}$.
2. Suppose that $\psi(x)$ is a non-zero, $\mathcal{C}^1$-function satisfying the identity

$$\psi(x)\psi(y) = \psi(x+y) \text{ for all } x, y \in \mathbb{R}. \tag{4}$$

   Show that $\psi(x) = e^{\lambda x}$ for some $\lambda \in \mathbb{C}$.
3. Prove that there is a positive number c so that

$$\int_{-\pi}^{\pi} |D_N(x)|dx \geq c\left(1 + \frac{1}{2} + \cdots + \frac{1}{N}\right) \geq c \log N. \tag{5}$$

4. Let f be a 2π-periodic function and set $f_y(x) = f(x+y)$. Express the Fourier coefficients of f_y in terms of those of f.
5. If f and g are 2π-periodic, then show that

$$f * g(x) = \int_{-\pi}^{\pi} f(y)g(x-y)dy \tag{6}$$

   is also 2π-periodic. How are the Fourier coefficients of $f * g$ related to those of f and g?

6. Compute the Fourier coefficients of the following functions:
   (a) The 2π-periodic extension of

$$f(x) = |x| \text{ for } x \in [-\pi, \pi]. \tag{7}$$

   (b) The 2π-periodic extension of

$$f(x) = \begin{cases} 1 \text{ for } x \in [0, \pi] \\ 0 \text{ for } x \in [-\pi, 0). \end{cases} \tag{8}$$

   (c) The 2π-periodic extension of

$$f(x) = x^2 \text{ for } x \in [-\pi, \pi]. \tag{9}$$