Math 509
Problem set 4, due February 13, 2018
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapters 10.2 and 10.3 in The Way of Analysis.

You should do the following problems, but you do not need to hand in your solutions:

1. Given a multi-index \( \alpha \) prove that \( |x^\alpha| \leq |x|^\alpha \). Hint: Use homogeneity, see Problem 4 below.

The following problems should be carefully written up and handed in.

1. Polar coordinates \((r, \theta)\) in the plane are defined by \( x = r \cos \theta, \ y = r \sin \theta \).

   Suppose that \( f \) is a \( C^2 \) function of \((x, y)\) and define \( g(r, \theta) = f(r \cos \theta, r \sin \theta) \).

   Show that
   \[
   \partial_r^2 f(r \cos \theta, r \sin \theta) + \partial_{\theta}^2 f(r \cos \theta, r \sin \theta) = \partial_r^2 g(r, \theta) + \frac{1}{r} \partial_r g(r, \theta) + \frac{1}{r^2} \partial_{\theta}^2 g(r, \theta),
   \]
   for all \((x, y) \neq (0, 0)\).

2. Let \( f \in C^{m+1}(\mathbb{R}^n) \), the \( m \)th order Taylor polynomial is defined to be
   \[
   T_m(y, x) = \sum_{j=0}^{m} \sum_{|\alpha| = j} \frac{\partial^\alpha f(y)}{\alpha!} (x - y)^\alpha.
   \]

   Prove that there is a point \( z \) on the line segment from \( x \) to \( y \) so that
   \[
   f(x) - T_m(y, x) = \sum_{|\alpha| = m+1} \frac{\partial^\alpha f(z)}{\alpha!} (x - y)^\alpha.
   \]

3. Prove the higher dimensional generalization of the binomial expansion
   \[
   \left( x_1 + \cdots + x_n \right)^k = \sum_{|\alpha| = k} \frac{x^\alpha}{\alpha!}.
   \]

   If \( \beta \) is a multi-index with \( |\beta| = k \), then what is \( \partial^\beta_x (x_1 + \cdots + x_n)^k \)? For a multi-index of length \( k \) prove that \( \frac{k!}{\beta!} \) is an integer.

4. A function defined in \( \mathbb{R}^n \setminus \{0\} \) is homogeneous of degree \( k \) if
   \[
   f(\lambda x_1, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n) \text{ for } \lambda \in (0, \infty).
   \]

   Show that \( f \) is \( C^1 \) and homogeneous of degree \( k \) if and only if
   \[
   \sum_{j=1}^{n} x_i \partial x_i f(x) = kf(x) \text{ for } x \in \mathbb{R}^n \setminus \{0\}.
   \]
5. Let \( R = [a, b] \times [c, d] \) be a bounded rectangle, and \( f \) a continuous function on \( R \). Prove that \( f \) is Riemann integrable on \( R \).

6. Let \( D \subset \mathbb{R}^2 \) be a closed subset. Recall that the boundary of \( D \), \( \partial D \), is defined by
   \[
   \partial D = \{ p \in \mathbb{R}^2 : \text{ for all } 0 < r, \quad B_r(p) \cap D \neq \emptyset \text{ and } B_r(p) \cap D^c \neq \emptyset \}\]
   Show that
   \[
   [\partial D]^c = D^c \cup \text{int } D.
   \]
   Recall that \( \text{int } D \) (the interior of \( D \)) is the largest open set contained in \( D \). If \( D \) is an open set, then \( \partial D = D \setminus D \). Can you find an example to show that \( \partial D \) and \( \partial D \) may not agree?

7. Suppose that \( R = [0, 1] \times [0, 1] \). Let \( f \) be a Riemann integrable function defined on \( R \). Let \( \{P_k \times Q_k\} \) denote the sequence of uniform partitions with side-lengths \( 1/2^k \). Show that for any choices of points \( Y^k = \{y_{ij}^{(k)}\} \) from these partitions we have that the Cauchy sums
   \[
   S(f; Y^k, P_k \times Q_k) = \sum_{1 \leq i, j \leq 2^k} \frac{f(y_{ij}^{(k)})}{2^{2k}}
   \]
   converge, as \( k \to \infty \), to \( \int_D f(x, y) dxdy \).