Math 509  
Problem set 4, due February 19, 2019  
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 12 in The Way of Analysis. The first two chapters of Stein and Shakarchi’s book on Fourier Analysis are another good reference for this material.

You should do the following problems, but you do not need to hand in your solutions:

1. Suppose that $f$ is a $2\pi$-periodic, Riemann integrable function defined on $\mathbb{R}$. Show that the integral

$$\int_x^{x+2\pi} f(y)\,dy$$

does not depend on $x$.

2. If $\langle f_k \rangle$ is a sequence of $2\pi$-periodic, continuous functions converging uniformly to a function $f$, then show that

$$\hat{f}(n) = \lim_{k \to \infty} \hat{f_k}(n) \text{ for all } n \in \mathbb{Z}.$$

The following problems should be carefully written up and handed in.

1. Let $f$ be a $2\pi$-periodic, Riemann integrable function.
   (a) Prove that $f$ is real valued if and only if

$$\hat{f}(n) = \overline{\hat{f}(-n)} \text{ for all } n \in \mathbb{Z}.$$  

(b) Prove that if $f$ is even ($f(x) = f(-x)$) then $\hat{f}(n) = \hat{f}(-n)$, for all $n \in \mathbb{Z}$.

(c) Show that if $f(x + \pi) = f(x)$, then $\hat{f}(2n + 1) = 0$, for all $n \in \mathbb{Z}$.

2. Suppose that $\psi(x)$ is a $C^1$-function satisfying the identity

$$\psi(x)\psi(y) = \psi(x + y) \text{ for all } x, y \in \mathbb{R}.$$ 

Show that $\psi(x) = e^{\lambda x}$ for some $\lambda \in \mathbb{C}$.

3. Prove that there is a positive number $c$ so that

$$\int_{-\pi}^{\pi} |D_N(x)|\,dx \geq c \left( 1 + \frac{1}{2} + \cdots + \frac{1}{N} \right) \geq c \log N.$$  

4. Let $f$ be a $2\pi$-periodic function and set $f_y(x) = f(x + y)$. Express the Fourier coefficients of $f_y$ in terms of those of $f$.

5. If $f$ and $g$ are $2\pi$-periodic, then show that

$$f \ast g(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,dy$$

is also $2\pi$-periodic. How are the Fourier coefficients of $f \ast g$ related to those of $f$ and $g$?
6. Compute the Fourier coefficients of the following functions:

(a) The $2\pi$-periodic extension of

\[ f(x) = |x| \text{ for } x \in [-\pi, \pi]. \tag{7} \]

(b) The $2\pi$-periodic extension of

\[ f(x) = \begin{cases} 
1 & \text{for } x \in [0, \pi] \\
0 & \text{for } x \in [-\pi, 0). 
\end{cases} \tag{8} \]

(c) The $2\pi$-periodic extension of

\[ f(x) = x^2 \text{ for } x \in [-\pi, \pi]. \tag{9} \]