Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

The following problems should be carefully written up and handed in.

1. Let $R = [a, b] \times [c, d]$ be a bounded rectangle, and $f$ a continuous function on $R$. Prove that $f$ is Riemann integrable on $R$.

2. Let $D \subset \mathbb{R}^2$ be a closed subset. Recall that the boundary of $D$, $\partial D$, is defined by $\partial D = \{ p \in \mathbb{R}^2 : \text{for all } 0 < r, \ B_r(p) \cap D \neq \emptyset \text{ and } B_r(p) \cap D^c \neq \emptyset \}$

Show that $[\partial D]^c = D^c \cup \text{int } D$.

Recall that $\text{int } D$ (the interior of $D$) is the largest open set contained in $D$. If $D$ is an open set, then show that $\partial D = \overline{D} \setminus D$.

Let $D = \{(x, y) : x^2 + y^2 < 1\} \setminus \{(x, 0) : -1 < x < 0\}$. What is $\partial D$? In this case do $\partial D$ and $\overline{D}$ agree?

3. Let $P, P'$ be partitions of $[0, 1]$, and suppose that the longest interval of $P'$ is shorter than the shortest interval of $P$. Suppose that $f(x, y)$ is a bounded function on $[0, 1] \times [0, 1]$. Show that

$$\text{Osc}(f, P' \times P') \leq 9 \text{Osc}(f, P \times P).$$

Hint: Look at the proof of the analogous 1-dimensional result.

4. Suppose that $R = [0, 1] \times [0, 1]$. Let $f$ be a Riemann integrable function defined on $R$. Let $\{P_k \times Q_k\}$ denote the sequence of uniform partitions with side-lengths $1/2^k$. Show that for any choices of points $Y^k = \{y^{(k)}_{ij}\}$ from these partitions we have that the Cauchy sums

$$S(f; Y^k, P_k \times Q_k) = \sum_{1 \leq i, j \leq 2^k} \frac{f(y^{(k)}_{ij})}{2^{2k}}$$

converge, as $k \to \infty$, to $\int_D f(x, y) \, dx \, dy$.

5. Let $f(x), g(x)$ be Riemann integrable functions on $[0, 1]$. Show that $h(x, y) = f(x)g(y)$ is a Riemann integrable function on $R = [0, 1] \times [0, 1]$ and that

$$\int_R h(x, y) \, dx \, dy = \int_0^1 f(x) \, dx \cdot \int_0^1 g(y) \, dy.$$