Math 509  
Problem set 7, due April 2, 2019  
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 10.1 and 10.2 in *The Way of Analysis* and the rest of Chapter 2 in *Calculus on Manifolds*.

You should do the following problem, but you do not need to hand in your solution:
1. Given a multi-index \( \alpha \) prove that \( |x^\alpha| \leq |x|^|\alpha| \). Hint: Use homogeneity.
2. Let \( A \) be a symmetric \( n \times n \) matrix. Prove that the quadratic form \( Q(x,y) = \langle Ax, y \rangle \) is non-degenerate if and only if all the eigenvalues of \( A \) are non-zero. Recall, the quadratic form is non-degenerate provided that for each \( x \neq 0 \), there is a vector \( y \) so that \( Q(x,y) \neq 0 \).

The following problems should be carefully written up and handed in.
1. Let \( f \in \mathcal{C}^{m+1}(\mathbb{R}^n) \), the \( m \)th order taylor polynomial is defined to be

\[
T_m(y, x) = \sum_{j=0}^{m} \sum_{\alpha: |\alpha| = j} \frac{\partial_x^\alpha f(y)}{\alpha!} (x - y)^\alpha.
\]

prove that there is a point \( z \) on the line segment from \( x \) to \( y \) so that

\[
f(x) - T_m(y, x) = \sum_{\alpha: |\alpha| = m+1} \frac{\partial_x^\alpha f(z)}{\alpha!} (x - y)^\alpha.
\]

2. Let \( f: \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
f(x, y) = \begin{cases} 
xy(x^2-y^2) & \text{for } (x, y) \neq (0, 0) \\
0 & \text{for } (x, y) = (0, 0).
\end{cases}
\]

Show that the partial derivatives \( \partial_x f, \partial_y f, \partial_x \partial_y f, \partial_y \partial_x f \) exist for all \( (x, y) \in \mathbb{R}^2 \), but

\[
\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0).
\]

3. Let \( \alpha \) be an \( n \)-multi-index, and let \( f \) and \( g \) be infinitely differentiable functions defined in \( \mathbb{R}^n \). Find a formula for \( \partial_x^\alpha (f \cdot g) \) in terms of partial derivatives of \( f \) and \( g \). Hint: Do the 1d-case first.

4. Let \( (x_j, y_j), j = 1, \ldots, n \), be pairs of real numbers, with \( x_i \neq x_j \) if \( i \neq j \). For each \( (a, b) \in \mathbb{R}^2 \) we define

\[
e(a, b) = \sum_{j=1}^{n} (y_j - (ax_j + b))^2.
\]
Find the pair \((a_0, b_0)\) that minimizes this function. Prove that the Hessian of \(e\) is positive definite at \((a_0, b_0)\).

5. Find and classify the critical points of the following functions:
   (a) \(f(x, y) = x^4 + x^2y^2 - y\).
   (b) \(f(x, y) = \frac{x}{1+x^2+y^2}\).
   (c) \(f(x, y) = x^4 + y^4 - x^3\).