Math 509
Problem set 7, due April 2, 2019
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 10.1 and 10.2 in The Way of Analysis and the rest of Chapter 2 in Calculus on Manifolds.

You should do the following problem, but you do not need to hand in your solution:

1. Given a multi-index $\alpha$ prove that $|x^\alpha| \leq |x|^\alpha$. Hint: Use homogeneity.
2. Let $A$ be a symmetric $n \times n$ matrix. Prove that the quadratic form $Q(x, y) = \langle Ax, y \rangle$ is non-degenerate if and only if all the eigenvalues of $A$ are non-zero. Recall, the quadratic form is non-degenerate provided that for each $x \neq 0$, there is a vector $y$ so that $Q(x, y) \neq 0$.

The following problems should be carefully written up and handed in.

1. Let $f \in C^{m+1}(\mathbb{R}^n)$, the $m$th order taylor polynomial is defined to be

$$T_m(y, x) = \sum_{j=0}^{m} \sum_{|\alpha| = j} \frac{\partial_x^\alpha f(y)}{\alpha!} (x - y)^\alpha.$$ 

prove that there is a point $z$ on the line segment from $x$ to $y$ so that

$$f(x) - T_m(y, x) = \sum_{|\alpha| = m+1} \frac{\partial_x^\alpha f(z)}{\alpha!} (x - y)^\alpha.$$ 

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$(1) \quad f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$$

Show that the partial derivatives $\partial_x f, \partial_y f, \partial_x \partial_y f, \partial_y \partial_x f$ exist for all $(x, y) \in \mathbb{R}^2$, but

$$(2) \quad \partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0).$$

3. Let $\alpha$ be an $n$-multi-index, and let $f$ and $g$ be infinitely differentiable functions defined in $\mathbb{R}^n$. Find a formula for $\partial_x^\alpha (f \cdot g)$ in terms of partial derivatives of $f$ and $g$. Hint: Do the $1d$-case first.

4. Let $(x_j, y_j), j = 1, \ldots, n$, be pairs of real numbers, with $x_i \neq x_j$ if $i \neq j$. For each $(a, b) \in \mathbb{R}^2$ we define

$$(3) \quad e(a, b) = \sum_{j=1}^{n} (y_j - (ax_j + b))^2.$$
5. Find the pair \((a_0, b_0)\) that minimizes this function. Prove that the Hessian of \(e\) is positive definite at \((a_0, b_0)\).

Find and classify the critical points of the following functions:

(a) \(f(x, y) = x^4 + x^2y^2 - y\).
(b) \(f(x, y) = \frac{x}{1+x^2+y^2}\).
(c) \(f(x, y) = x^4 + y^4 - x^3\).