Math 509
Problem set 7, due April 2, 2019
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 10.1 and 10.2 in **The Way of Analysis** and the rest of Chapter 2 in **Calculus on Manifolds**.

You should do the following problem, but you do not need to hand in your solution:

1. Given a multi-index α prove that $|x^{\alpha}| \leq |x|^{|\alpha|}$. Hint: Use homogeneity.
2. Let A be a symmetric $n \times n$ matrix. Prove that the quadratic form $Q(x, y) = \langle Ax, y \rangle$ is non-degenerate if and only if all the eigenvalues of A are non-zero. Recall, the quadratic form is non-degenerate provided that for each $x \neq 0$, there is a vector y so that $Q(x, y) \neq 0$.

The following problems should be carefully written up and handed in.

1. Let $f \in \mathcal{C}^{m+1}(\mathbb{R}^n)$, the mth order taylor polynomial is defined to be
$$T_m(y, x) = \sum_{j=0}^{m} \sum_{\{\alpha : |\alpha| = j\}} \frac{\partial_x^{\alpha} f(y)}{\alpha!}(x - y)^{\alpha}.$$
prove that there is a point z on the line segment from x to y so that
$$f(x) - T_m(y, x) = \sum_{\{\alpha : |\alpha| = m+1\}} \frac{\partial_x^{\alpha} f(z)}{\alpha!}(x - y)^{\alpha}.$$
2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0) \\ 0 & \text{for } (x, y) = (0, 0). \end{cases} \tag{1}$$
Show that the partial derivatives $\partial_x f, \partial_y f, \partial_x \partial_y f, \partial_y \partial_x f$ exist for all $(x, y) \in \mathbb{R}^2$, but
$$\partial_x \partial_y f(0, 0) \neq \partial_y \partial_x f(0, 0). \tag{2}$$
3. Let α be an n-multi-index, and let f and g be infinitely differentiable functions defined in $\mathbb{R}^n$. Find a formula for $\partial_x^{\alpha}(f \cdot g)$ in terms of partial derivatives of f and g. Hint: Do the $1d$-case first.
4. Let (x_j, y_j), $j = 1, \ldots, n$, be pairs of real numbers, with $x_i \neq x_j$ if $i \neq j$. For each $(a, b) \in \mathbb{R}^2$ we define
$$e(a, b) = \sum_{j=1}^{n} (y_j - (ax_j + b))^2. \tag{3}$$

Find the pair (a_0, b_0) that minimizes this function. Prove that the Hessian of e is positive definite at (a_0, b_0).

5. Find and classify the critical points of the following functions:
   (a) $f(x, y) = x^4 + x^2y^2 - y$.
   (b) $f(x, y) = \frac{x}{1+x^2+y^2}$.
   (c) $f(x, y) = x^4 + y^4 - x^3$.