Math 509  
Problem set 8, due April 9, 2019  
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 3 in *Calculus on Manifolds*.

You should do the following problems, but you do not need to hand in your solution:

1. Let \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) be defined by

\[
 f(x) = \begin{cases} 
 1 & \text{for } x < y \\
 0 & \text{for } x \geq y.
\end{cases}
\]

Show that \( f \) is integrable and compute

\[
\int_{[0,1] \times [0,1]} f(x, y) \, dx \, dy,
\]

using the definition.

2. Let \( f, g \) be Riemann integrable functions on \([0, 1] \times [0, 1]\). Show that the functions \( f + g, f \cdot g, \max\{f, g\}, \min\{f, g\} \) are also Riemann integrable on \([0, 1] \times [0, 1]\).

3. Show that if \( f \) is Riemann integrable on \([0, 1] \times [0, 1]\), then \(|f|\) is as well and

\[
\left| \int_{[0,1] \times [0,1]} f(x, y) \, dx \, dy \right| \leq \int_{[0,1] \times [0,1]} |f(x, y)| \, dx \, dy.
\]

The following problems should be carefully written up and handed in.

1. Let \( f : [0, 1] \times [0, 1] \to \mathbb{R} \) be defined by

\[
 f(x) = \begin{cases} 
 0 & \text{for } x \text{ irrational,} \\
 0 & \text{for } x \text{ rational and } y \text{ irrational}, \\
 \frac{1}{q} & \text{for } x \text{ rational and } y = \frac{p}{q}.
\end{cases}
\]

Show that \( f \) is integrable and compute

\[
\int_{[0,1] \times [0,1]} f(x, y) \, dx \, dy.
\]

2. Let \( P, P', Q, Q' \) be partitions of \([0, 1]\). Assume that \(|P'|\) is smaller than the shortest interval in \( P \) and that \(|Q'|\) is smaller than the shortest interval in \( Q \). If \( f \) is a bounded function on \([0, 1] \times [0, 1]\), then show that

\[
S^+(f, P' \times Q') - S^-(f, P' \times Q') \leq 9[S^+(f, P \times Q) - S^-(f, P \times Q)].
\]

3. Let \( f \) be a continuous function in \([0, 1] \times [0, 1]\); prove that

\[
\int_{0}^{1} \int_{0}^{y} f(x, y) \, dx \, dy = \int_{0}^{1} \int_{x}^{1} f(x, y) \, dy \, dx.
\]
4. Let \( g_1, g_2 : \mathbb{R}^2 \to \mathbb{R} \) be continuously differentiable functions such that \( \partial_y g_1 = \partial_x g_2 \). Define the function

\[
\tag{8}
f(x, y) = \int_0^x g_1(t, 0)\,dt + \int_0^y g_2(x, t)\,dt.
\]

Show that \( f \) is continuously differentiable and that \( \partial_x f(x, y) = g_1(x, y) \). What is \( \partial_y f(x, y) \)? Can you find another formula for \( f \)?

5. Let \( A, B \subset \mathbb{R}^2 \) be Riemann measurable, and assume that for each \( c \in \mathbb{R} \) the sets

\[
\tag{9}
A_c = \{x : (x, c) \in A\} \quad \text{and} \quad B_c = \{x : (x, c) \in B\}
\]

are also Riemann measurable. Suppose that for each \( c \in \mathbb{R} \) we know that \( |A_c| = |B_c| \). Prove that \( |A| = |B| \). That is: if the length of each slice, \( A_c \), equals that of the corresponding \( B_c \), then the area of \( A \) equals the area of \( B \). Use this to give a geometric proof that the area of triangle is \( \frac{1}{2}bh \), where \( b \) is the length of the base of the triangle and \( h \) is the height, see Figure 1.

\[\text{Figure 1. Triangle} \]