Math 509  
Problem set 8, due March 27, 2018  
Dr. Epstein

In Bak & Newman read Chapters 1, 2, and 3. Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For $z = x + iy$, we define the functions

$$\Re(z) = x, \ \Im(z) = y, \ \bar{z} = x - iy.$$  

You should do the following problems, but you do not need to hand in your solutions:

1. Show that $\{z : |z - w| < r\}$ is a disk of radius $r$ centered at $w$.
2. Geometrically what is the set
   $$\{z : a|z|^2 + b\Re(z^2) = 1\}?$$
   Here $a$ and $b$ are real numbers with $0 < a$. Hint: What are $|z + \bar{z}|^2$ and $|z - \bar{z}|^2$?
3. If $z = \frac{1+i\sqrt{3}}{2}$, then what are $z^2$ and $z^{-1}$?
4. For $w$ a non-zero complex number, geometrically what is the set
   $$\{z : 0 < \arg(z\bar{w}) < \frac{\pi}{2}\}?$$

The following problems should be carefully written up and handed in.

1. Give complete proofs that for $z_1, z_2 \in \mathbb{C}$
   $$|z_1 + z_2| \leq |z_1| + |z_2|$$
   and interpret this result geometrically.
2. For $z_1, z_2 \in \mathbb{C}$ prove that
   $$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2),$$
   and give a simple geometric description of the set $\{z : \Re(z\bar{w}) > 0\}$.
3. Suppose that $P(z)$ is a polynomial with real coefficients. Show that
   $$P(\bar{z}) = \overline{P(z)}.$$  
   Conclude that if $z_0$ is a root of $P$, then so is $\bar{z}_0$.
   Show that if $f(z)$ is given by a power series with real coefficients, convergent in $D_R(0)$, for an $0 < R$, then $f(\bar{z}) = \overline{f(z)}$. Show that a convergent power series has real coefficients if and only if $f(x) \in \mathbb{R}$, for $x \in \mathbb{R}$ with $|x| < R$.
4. Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where $a_n \geq a_{n-1} \geq \cdots \geq a_0 \geq 0$ are real numbers. By considering the polynomial $(1 - z)P(z)$ show that all solutions of $P(z) = 0$ lie inside the closed unit disk. Hint: Use the triangle inequality.
6. Show that any complex-valued polynomial in two real variables,
\[ P(x, y) = \sum_{0 \leq m, n \leq N} a_{mn} x^m y^n, \]
can be re-expressed in the form
\[ P = \sum_{0 \leq m, n \leq N} a_{mn} z^m \bar{z}^n. \]
Show that \( P \) has a complex derivative if and only if \( a_{mn} = 0 \) for \( 0 < n \).

7. Suppose that \( f \) is a complex differentiable function defined in a connected open set \( D \subset \mathbb{C} \). Show that if \( f(z) \in \mathbb{R} \) for all \( z \in D \), then \( f \) is a constant function.