Math 509
Problem set 8, due April 9, 2019
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 3 in **Calculus on Manifolds**.

You should do the following problems, but you do not need to hand in your solution:

1. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 \text{ for } x < y \\ 0 \text{ for } x \geq y. \end{cases} \tag{1}$$

Show that f is integrable and compute

$$\int_{[0,1]\times[0,1]} f(x,y)dxdy, \tag{2}$$

using the definition.

2. Let f, g be Riemann integrable functions on $[0,1] \times [0,1]$. Show that the functions $f + g$, $f \cdot g$, $\max\{f, g\}$, $\min\{f, g\}$ are also Riemann integrable on $[0,1] \times [0,1]$.
3. Show that if f is Riemann integrable on $[0,1] \times [0,1]$, then $|f|$ is as well and

$$\left| \int_{[0,1]\times[0,1]} f(x,y)dxdy \right| \leq \int_{[0,1]\times[0,1]} |f(x,y)|dxdy. \tag{3}$$

The following problems should be carefully written up and handed in.

1. Let $f : [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 \text{ for } x \text{ irrational}, \\ 0 \text{ for } x \text{ rational and } y \text{ irrational}, \\ \frac{1}{q} \text{ for } x \text{ rational and } y = \frac{p}{q}. \end{cases} \tag{4}$$

Show that f is integrable and compute

$$\int_{[0,1]\times[0,1]} f(x,y)dxdy. \tag{5}$$

2. Let P, P', Q, Q' be partitions of $[0,1]$. Assume that $|P'|$ is smaller than the shortest interval in P and that $|Q'|$ is smaller than the shortest interval in Q. If f is a bounded function on $[0,1] \times [0,1]$, then show that

$$S^+(f, P' \times Q') - S^-(f, P' \times Q') \leq 9[S^+(f, P \times Q) - S^-(f, P \times Q)]. \tag{6}$$

3. Let f be a continuous function in $[0,1] \times [0,1]$; prove that

$$\int_0^1 \int_0^y f(x,y)dxdy = \int_0^1 \int_x^1 f(x,y)dydx. \tag{7}$$

4. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable functions such and that $\partial_y g_1 = \partial_x g_2$. Define the function

$$f(x, y) = \int_0^x g_1(t, 0)dt + \int_0^y g_2(x, t)dt. \tag{8}$$

Show that f is continuously differentiable and that $\partial_x f(x, y) = g_1(x, y)$. What is $\partial_y f(x, y)$? Can you find another formula for f?

5. Let $A, B \subset \mathbb{R}^2$ be Riemann measurable, and assume that for each $c \in \mathbb{R}$ the sets

$$A_c = \{x : (x, c) \in A\} \text{ and } B_c = \{x : (x, c) \in B\} \tag{9}$$

are also Riemann measurable. Suppose that for each $c \in \mathbb{R}$ we know that $|A_c| = |B_c|$. Prove that $|A| = |B|$. That is: if the length of each slice, A_c, equals that of the corresponding B_c, then the area of A equals the area of B. Use this to give a *geometric* proof that the area of triangle is $\frac{1}{2}bh$, where b is the length of the base of the triangle and h is the height, see Figure 1.

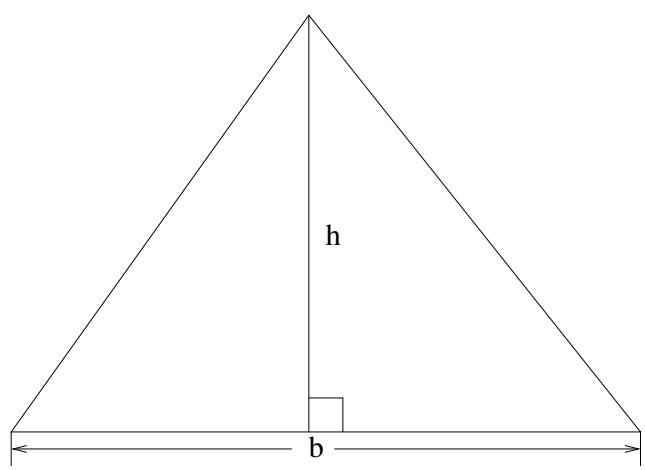

FIGURE 1. Triangle