In Bak & Newman read Chapters 4, 5.1, 6.1. Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For $z = x + iy$, we define the functions

$\Re(z) = x, \ \Im(z) = y, \ \bar{z} = x - iy$.

You should do the following problems, but you do not need to hand in your solutions:

1. Prove the familiar trigonometric identities

$$\sin(z + w) = \sin(z) \cos(w) + \sin(w) \cos(z) \text{ and}$$

$$\cos(z + w) = \cos(z) \cos(w) - \sin(z) \sin(w).$$

Explain why they actually hold for all complex numbers $z, w$.

2. Prove that for $k \in \mathbb{Z} \setminus \{-1\}$

$$\int_C z^k \, dz = 0$$

in two ways: first by a direct calculation of the contour integral, and second by showing that $z^k$ is the complex derivative of an analytic function. For $k \geq 0$, there is a third way, which does not work for $k < 0$. What is it?

The following problems should be carefully written up and handed in.

1. (a) Let $f$ be a complex valued, $C^1$-function defined in an open set $D \subset \mathbb{C}$. Show that

$$df = \partial_z f \, dz + \partial_{\bar{z}} f \, d\bar{z},$$

where recall that

$$\partial_z = \frac{1}{2}(\partial_x - i \partial_y), \ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i \partial_y),$$

$$dz = dx + idy, \ d\bar{z} = dx - idy,$$

and $df$ is defined by: $df = \partial_x f \, dx + \partial_y f \, dy$.

(b) With $\partial_z$ and $\partial_{\bar{z}}$ as defined above show that

$$\partial_{\bar{z}} \partial_z u = \partial_z \partial_{\bar{z}} u = \frac{1}{4}(\partial_x^2 + \partial_y^2)u = \frac{1}{4} \Delta u.$$

$\Delta$ is called the Laplace operator.

2. (a) Show that if $f = u + iv$ is an analytic function in a disk $D_r(w)$, then

$$\Delta u = \Delta v = 0.$$

You should explain why this second order equation makes sense for any analytic function. Show that there is no analytic function $f(x, y) = u(x, y) + iv(x, y)$ with $u(x, y) = x^2 + y^2$. 

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(b) For an analytic function $u + iv$, show that
\[
dv = -u_y dx + u_x dy.
\]

3. Let $f$ be a continuous complex valued function defined on a $C^1$-arc, $\gamma : [0, 1] \to \mathbb{C}$. If $\Gamma = \gamma([0, 1])$, as an oriented arc, then prove that
\[
\left| \int_{\Gamma} f dz \right| \leq \int_{\Gamma} |f(z)||dz|.
\]
Along $\Gamma$ we define $|dz| = |\gamma'(t)| dt$ to be arclength measure.

4. Suppose that $f$ is a complex valued function defined on the interval $(-1, 1)$, and assume that there is a constant $M$ so that the derivatives of $f$ satisfy
\[
|f^{(j)}(y)| \leq M j! \text{ for } |y| < 1, \text{ and } j = 0, 1, \ldots
\]
(a) Prove that, for $x \in (-1, 1)$, $f(x)$ is represented by its Taylor series, that is:
\[
f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(0)}{j!} x^j.
\]
Hint: It is not enough to show that the power series converges.

(b) Show that this power series defines an analytic function in $\{z : |z| < 1\}$. How is this function related to $f$?

(c) By considering the function
\[
g(x) = \begin{cases} 
    e^{-\frac{1}{x^2}} \text{ for } x \neq 0 \\
    0 \text{ for } x = 0,
\end{cases}
\]
show that no estimate on the growth of the derivatives at a $x = 0$,
\[
\{|g^{(j)}(0)| : j = 0, 1, \ldots \},
\]
suffices to show that $g(x)$ is represented by its Taylor series in a non-trivial interval around 0.

5. Suppose that $f$ is analytic in a convex region $D \subset \mathbb{C}$.

(a) If $\gamma = (\gamma_1(t) + i \gamma_2(t)) : [0, 1] \to D$ is a $C^1$-curve, then prove that
\[
\frac{d}{dt} f(\gamma(t)) = f'(\gamma(t)) \gamma'(t).
\]

(b) If $|f'(z)| < 1$, for all $z \in D$, then show that
\[
|f(z) - f(w)| < |z - w|, \text{ for all } z, w \in D.
\]