Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 3 in *Calculus on Manifolds*.

You should do the following problem, but you do not need to hand in your solution:

1. Let $R_\theta$ be the rotation of the plane through the angle $0 < \theta < 2\pi$:

\[
R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}.
\]

If $D = [a, b] \times [c, d]$ is a rectangle show that the area of $R_\theta D$ is equal to that of $D$. Show more generally that if $D$ is a Riemann measurable subset of $\mathbb{R}^2$, then so is $R_\theta(D)$, and

\[
|D| = |R_\theta(D)|.
\]

The following problems should be carefully written up and handed in.

1. Let $D$ be a Riemann measurable region in the plane and $< f_n(x, y) >$ a sequence of Riemann integrable functions defined in $D$. Show that if $< f_n >$ converges uniformly to $f$, then $f$ is Riemann integrable and

\[
\lim_{n \to \infty} \int_D f_n(x, y) \, dx \, dy = \int_D f(x, y) \, dx \, dy.
\]

2. Suppose that you are told that the circumference of a circle of radius 1 is $2\pi$. Show that the area of the unit disk satisfies

\[
\int_{\{(x, y): x^2 + y^2 < 1\}} \, dx \, dy = \pi.
\]

What is the area of the disk of radius $0 < r$?

3. Let $f(x)$ be a non-negative, $C^1$-function defined on $[a, b]$ and set

\[D_f = \{(x, y) : a \leq x \leq b \text{ and } 0 \leq y \leq f(x)\}.
\]

Show that if $g$ is a Riemann Integrable function defined on $D_f$, then

\[
\int_{D_f} g(x, y) \, dx \, dy = \int_{[a, b] \times [0, 1]} g(s, tf(s)) f(s) \, ds \, dt.
\]

Assuming that $g$ is continuous and defined on an open set containing $D_f$, explain how to prove this for $f \in C^0([a, b])$ by using an approximation argument.

4. Let $D_1, D_2$ be closed, Riemann measurable sets such that $\text{int } D_1 \cap \text{int } D_2 = \emptyset$. Show that $|D_1 \cup D_2| = |D_1| + |D_2|$.
5. Let \( f(\theta) \) be a continuous, positive function defined on \([0, 2\pi]\), such that \( f(0) = f(2\pi) \), and set

\[
C_f = \{(r \cos \theta, r \sin \theta) : 0 \leq r \leq f(\theta), \text{ for } 0 \leq \theta \leq 2\pi\}.
\]

Prove that \( C_f \) is a Riemann measurable set, and

\[
|C_f| = \int_0^{2\pi} \frac{[f(\theta)]^2}{2} \, d\theta.
\]

Hint: First prove this assuming the \( f \) is \( C^1 \), and then for \( f \) in \( C^0 \).