Math 509
Problem set 9, due April 16, 2019
Dr. Epstein

Your solutions to these problems should be written in English: Use complete sentences and paragraphs.

For this week, read Chapter 3 in **Calculus on Manifolds**.

You should do the following problem, but you do not need to hand in your solution:

1. Let R_θ be the rotation of the plane through the angle $0 < \theta < 2\pi$:

$$R_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1}$$

   If $D = [a, b] \times [c, d]$ is a rectangle show that the area of $R_\theta D$ is equal to that of D. Show more generally that if D is a Riemann measurable subset of $\mathbb{R}^2$, then so is $R_\theta(D)$, and

$$|D| = |R_\theta(D)|. \tag{2}$$

The following problems should be carefully written up and handed in.

1. Let D be a Riemann measurable region in the plane and $< f_n(x, y) >$ a sequence of Riemann integrable functions defined in D. Show that if $< f_n >$ converges uniformly to f, then f is Riemann integrable and

$$\lim_{n\to\infty} \int_D f_n(x, y)dxdy = \int_D f(x, y)dxdy.$$

2. Suppose that you are told that the circumference of a circle of radius 1 is 2π. Show that the area of the unit disk satisfies.

$$\int\limits_{\{(x,y):\, x^2+y^2<1\}} dxdy = \pi.$$

   What is the area of the disk of radius $0 < r$?

3. Let $f(x)$ be a non-negative, $\mathcal{C}^1$-function defined on $[a, b]$ and set

$$D_f = \{(x, y) : \ a \le x \le b \text{ and } 0 \le y \le f(x)\}.$$

   Show that if g is a Riemann Integrable function defined on D_f, then

$$\int_{D_f} g(x, y)dxdy = \int\limits_{[a,b]\times[0,1]} g(s, tf(s))f(s)dsdt.$$

   Assuming that g is continuous and defined on an open set containing D_f, explain how to prove this for $f \in \mathcal{C}^0([a, b])$ by using an approximation argument.

4. Let D_1, D_2 be closed, Riemann measurable sets such that $\operatorname{int} D_1 \cap \operatorname{int} D_2 = \emptyset$. Show that $|D_1 \cup D_2| = |D_1| + |D_2|$.

5. Let $f(\theta)$ be a continuous, positive function defined on $[0, 2\pi]$, such that $f(0) = f(2\pi)$, and set

$$C_f = \{(r\cos\theta, r\sin\theta) : \ 0 \le r \le f(\theta), \ \text{for } 0 \le \theta \le 2\pi\}. \tag{3}$$

Prove that C_f is a Riemann measurable set, and

$$|C_f| = \int_0^{2\pi} \frac{[f(\theta)]^2}{2} d\theta. \tag{4}$$

Hint: First prove this assuming the f is $\mathcal{C}^1$, and then for f in $\mathcal{C}^0$.