Reading:  Chapter 3 of Stein-Shakarchi.

Homework assignment:  The solutions to the following problems should be carefully written up and handed in.

2. Stein and Shakarchi page 106, problem 17.
5. If \( c(t) = x(t) + iy(t) \) is a \( C^1 \)-mapping from \([a, b]\) to \( \mathbb{C} \setminus \{0\} \), then show that we can express \( c(t) = r(t)e^{i\theta(t)} \), where \( r \) and \( \theta \) are \( C^1 \)-functions.
6. Suppose that \( f \) is a holomorphic function in \( \Omega \). Prove that \( \log f \) can be defined as a single valued holomorphic function in \( \Omega \), if and only if \( f^{\frac{1}{n}} \) has a single valued holomorphic branch, in \( \Omega \), for every \( n \in \mathbb{N} \).
7. Show, by directly computing the kernel and the co-kernel, that the index of the operator \( T_{z^n} \) equals \(-n\), for all \( n \in \mathbb{Z} \).
8. Let \( \Pi_+ : L^2(S^1) \to H^2(S^2) \) be the orthogonal projection onto the boundary values of functions holomorphic in \( D_1 \), and set
   \[
   f(z) = \sum_{j=-N}^{N} a_j z^j.
   \]  
   (1)
   Show that the operator \( [\Pi_+, f] \) has finite rank.
9. Let \( f \) be a holomorphic function defined in a neighborhood of \( D_1(0) \), non-vanishing on \( |z| = 1 \), with \( m \) zeros in \( D_1(0) \). By directly computing the kernel and co-kernel, show that the index \( T_f \) equals \(-m\).
10. Find the kernel function of the orthogonal projection $B : L^2(D_1) \to H^2(D_1)$, that is, the function $b(z, w)$ defined on $D_1 \times D_1$, so that

$$Bf(z) = \int_{D_1} b(z, w) f(w) dxdy. \quad (2)$$

Recall that $H^2(D_1)$ are the holomorphic functions on $D_1$, which are square integrable. Hint: It’s easy to find an orthonormal basis for $H^2(D_1)$. 