Reading: Taylor sections 2.1, 2.2, 2.4

1. The function $G_2(x, y) = [2 \pi]^{-1} \log \|x - y\|$ is defined for $x \neq y$ in $\mathbb{R}^2$. Show that if $\varphi \in C^2(\mathbb{R}^2)$ has compact support, then

$$u(x) = \int_{\mathbb{R}^2} G_2(x, y) \varphi(y) dy$$

is a $C^2$-function satisfying $\Delta u = \varphi$. Show that limit

$$\gamma = \lim_{x \to \infty} \frac{u(x)}{\log r}$$

eexists and give a formula for its value. Under what conditions does $\gamma = 0$?

2. Let $r_\theta$ be the rotation of $\mathbb{R}^2$ defined by the matrix

$$r_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$ 

We let $R_\theta$ denote the operator defined by $(R_\theta u)(x) = u(r_\theta x)$.

(a) Show that $\Delta \circ R_\theta = R_\theta \circ \Delta$.

(b) Show that if $u$ is a harmonic function defined in $B_R(0)$ then

$$v(x) = \frac{1}{2\pi} \int_0^{2\pi} u(r_\theta x) d\theta$$

is harmonic in $B_R(0)$ and rotationally invariant, that is $R_\theta v = v$ for any $\theta$.

(c) Give a formula for $v$ (which you must prove) and use it to give a new proof of the mean value property for harmonic functions.

3. Show that if $f$ is a holomorphic function in $\Omega \subset \mathbb{R}^2$, then $u = \log |f|$ satisfies the mean value inequality

$$u(x) \leq \frac{1}{\pi r^2} \iint_{B_r(x)} u dA,$$

provided $\overline{B_r(x)} \subset \Omega$. Show that if $f^{-1}(\{0\}) \neq \emptyset$, then the inequality is sometimes strict.
4. Show that if \( f \) is a \( C^2 \)-function defined in an open subset of \( \mathbb{R}^n \), which satisfies the mean value property, then \( \Delta f = 0 \).

5. Exercise 3 (a)–(c) on page 139 of Taylor.

6. Show that, for \( n \geq 3 \), there is a constant \( c_n \), so that

\[
g_n(x) = \frac{c_n}{\|x\|^{n-2}}
\]

is a fundamental solution for the Laplace operator on \( \mathbb{R}^n \). That is, we define the operator

\[
G_nf(x) = \int_{\mathbb{R}^n} g_n(x - y) f(y) dy;
\]

if \( f \in C_c^2(\mathbb{R}^n) \), then \( \Delta \circ G_n f = G_n \circ \Delta f = f \). Describe \( c_n \) geometrically. Prove that for every \( f \in C_c^2(\mathbb{R}^n) \) the equation \( \Delta u = f \) has a unique, bounded solution.

7. Consider the operator \( L = \Delta + c^2 \) acting on functions defined in \( \mathbb{R}^3 \). Here \( c \in \mathbb{R} \).

   (a) Find all solutions to the equation

\[
Lu = 0
\]

with spherical symmetry, that is depending only on \( \|x\| \).

(b) Prove that

\[
k(x,y) = -\frac{\cos(c\|x-y\|)}{4\pi\|x-y\|}
\]

is a fundamental solution for \( L \); that is if we define

\[
Kf(x) = \int_{\mathbb{R}^3} k(x,y)f(y) dy,
\]

then for \( f \in C_c^2(\mathbb{R}^3) \) we have

\[
K \circ Lf = L \circ Kf = f.
\]
8. Assume that in a neighborhood of \((x_0, y_0) \in \mathbb{R}^2\), \(u(x, y)\) is a smooth solution to the quasi-linear PDE:

\[
a(u_x, u_y)u_{xx} + 2b(u_x, u_y)u_{xy} + c(u_x, u_y)u_{yy} = 0.
\]

Suppose that in some neighborhood of \((x_0, y_0)\) the variables

\[
\xi = u_x(x, y) \quad \eta = u_y(x, y)
\]

define local coordinates; set \(\phi = xu_x + yu_y - u\). Prove that, as a function of \((\xi, \eta)\), \(\phi\) satisfies

\[
x = \phi_\xi, \quad y = \phi_\eta
\]

and the linear PDE:

\[
a(\xi, \eta)\phi_{\eta\eta} - 2b(\xi, \eta)\phi_{\xi\eta} + c(\xi, \eta)\phi_{\xi\xi} = 0.
\]