

Subfactors from Penn on.

Vaughan Jones,
Vanderbilt

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Cohomological invariants for groups of outer automorphisms algebras

COLIN E. SUTHERLAND

Automorphism groups and invariant states

ERLING STØRMER

Compact ergodic groups of automorphisms

MAGNUS B. LANDSTAD

Actions of discrete groups on factors

V. F. R. JONES

Ergodic theory and von Neumann algebras

CALVIN C. MOORE

Topologies on measured groupoids

ARLAN RAMSAY

The Seminar



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For the purposes of this talk a II_1 factor M should be thought of as a "field" (albeit noncommutative and with a ton of zero-divisors).

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We see that all dimensions in this case are given by the numbers $\frac{k}{n}$ where k is a non-negative integer (or infinity).

Thus the II_1 factors are infinite dimensional versions of $M_n(\mathbb{C})$ the dimensions of whose modules "fill in the gaps" of those of $M = M_n(\mathbb{C})$.

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If you want an example you can take the group $PSL(2, \mathbb{Z}) < PSL_2(\mathbb{R})$. The "holomorphic discrete series" of $PSL_2(\mathbb{R})$ is the natural unitary action of $PSL_2(\mathbb{R})$ on L^2 holomorphic functions on the upper half plane with measure $\frac{dx dy}{y^{2-n}}$.

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Radulescu.

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Picture:



1

2

3

4

$$2.6180339\dots = 4 \cos^2 \pi/5$$

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Which possesses a trace functional tr completely defined by the formula

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I will split the talk up somewhat artificially into three parts:

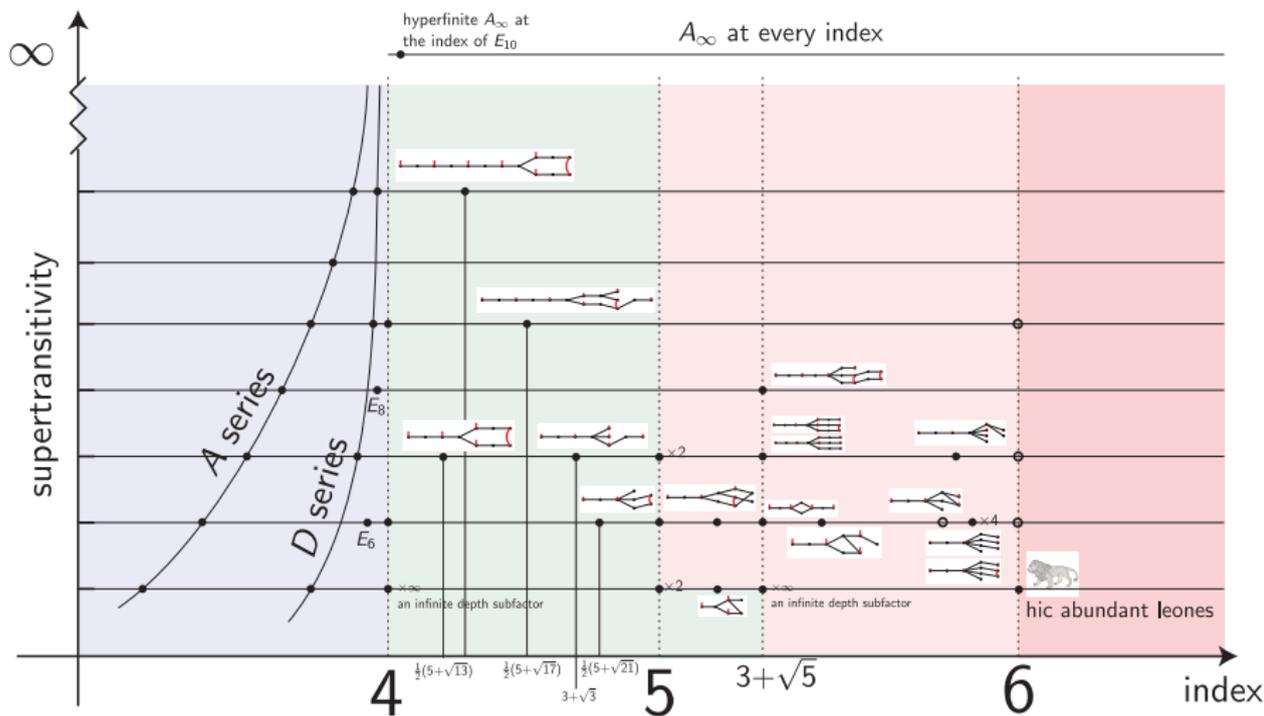
The internal theory of subfactors.

Interactions with other parts of mathematics.

Interactions with physics.

Internal theory:

One may sum up the internal theory developments with the following picture:



Ocneanu Popa Wenzl Goodman de la Harpe Haagerup Asaeda Izumi Bisch
Graham Lehrer Morrison Peters Snyder Bigelow Xu Grossman Liu Penneys
Tener Evans Gannon Etingof Nikschych Ostrik Yasuda Calegari Jobs
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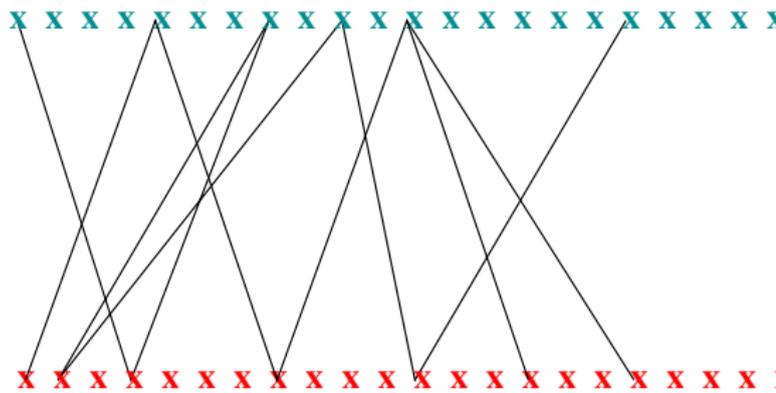
$${}_M H_N$$

.

It can be quite confusing because for a subfactor $N \subseteq M$, M is an $N - M$, $M - M$, $M - N$ and $N - N$ bimodule whereas typically in bimodule theory one considers a single ${}_M H_M$ and takes the tensor powers over M -
 $\otimes_M^k H$.

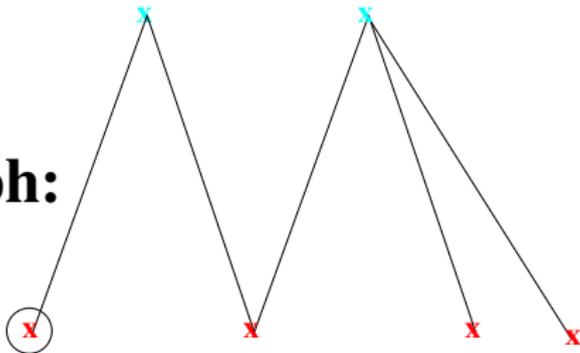
N–M bimodules

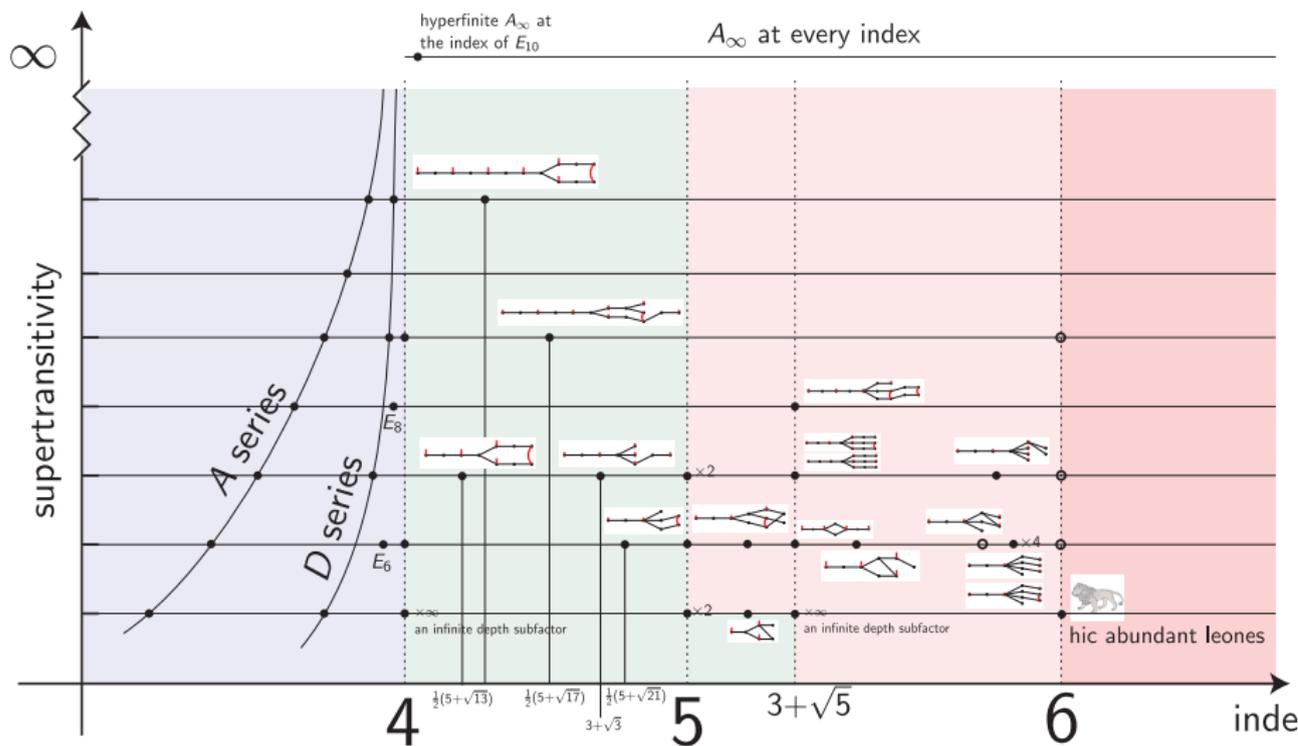
induction/restriction



N–N bimodules

The principal graph:





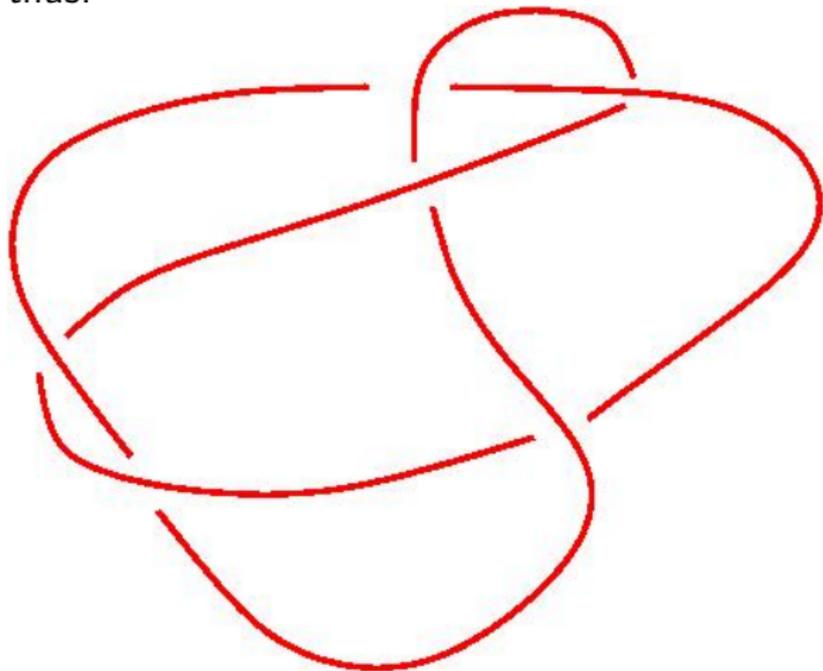
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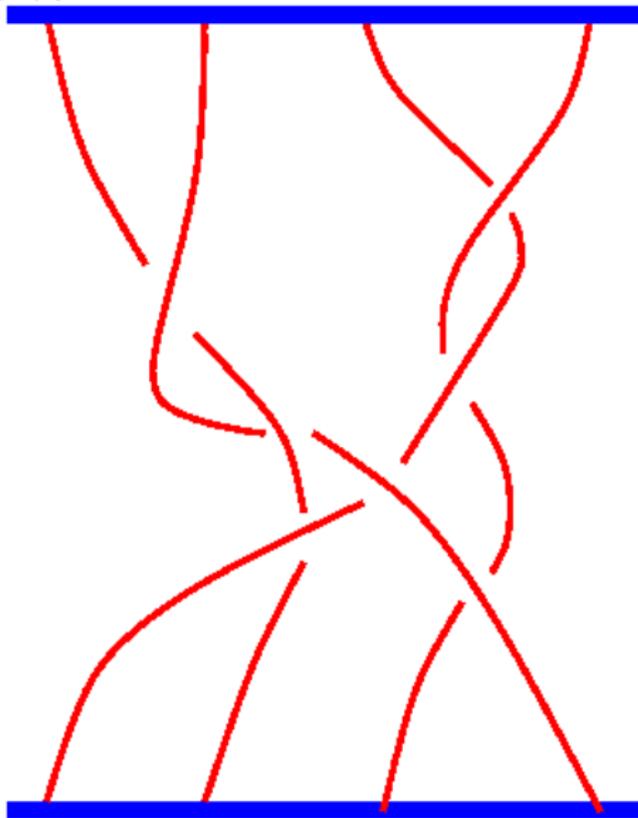
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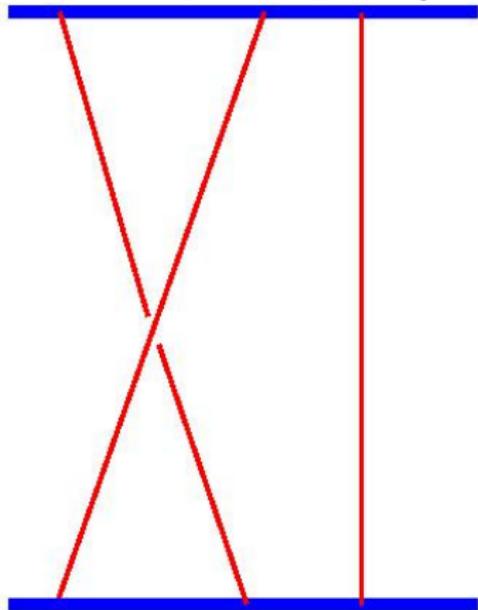


Braids are collections of curves in 3 dimensional space connecting two bars thus:

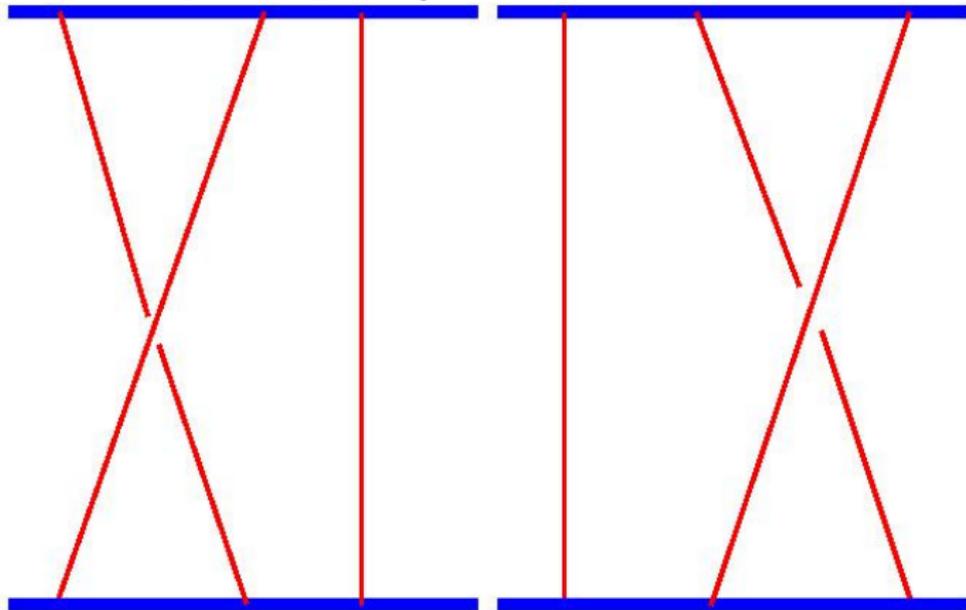


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The presenting relations are

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

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Represent B_n in II_1 factor by

$$\sigma_i \rightarrow t e_1 - (1 - e_i)$$

with $[M : N] = 2 + t + t^{-1}$.

The connection between braids and links:

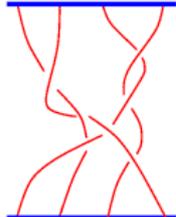
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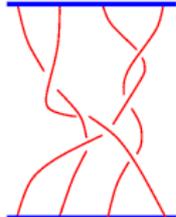
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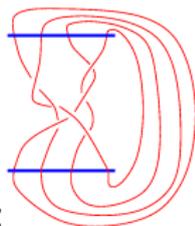
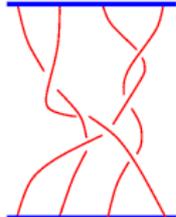


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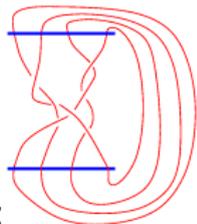
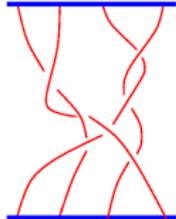


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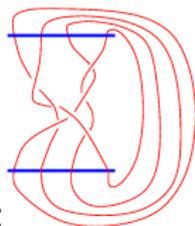
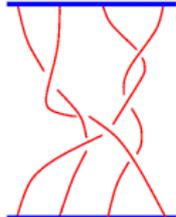
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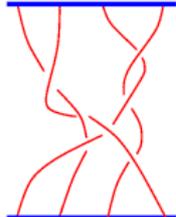


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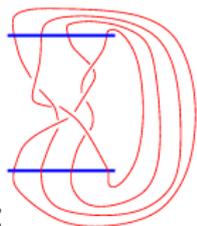
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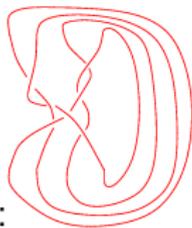
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A LINK!

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e.g. for trefoil knot

$$V_K(t) = t + t^3 - t^4$$

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- (i) Take L and represent it as the closure of a braid α .
- (ii) Represent the braid α in the algebra of the e_i 's via $\sigma_i \mapsto te_i - (1 - e_i)$
- (iii) Take the trace of the braid in the algebra, multiply by a simple fudge factor and voilà.

e.g. for trefoil knot

$$V_K(t) = t + t^3 - t^4$$

So 1984 was a bad year for Orwell characters, a great year for me (at Penn)!

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The understanding of the structure and even the mathematical nature of these invariants is an ongoing body of research with strong interactions with physics, in particular conformal field theory.

(Kontsevich, Bar Natan, Lawrence, Le, Ohtsuki, Garoufalidis, Aganagic, Vafa, Gukov.....)

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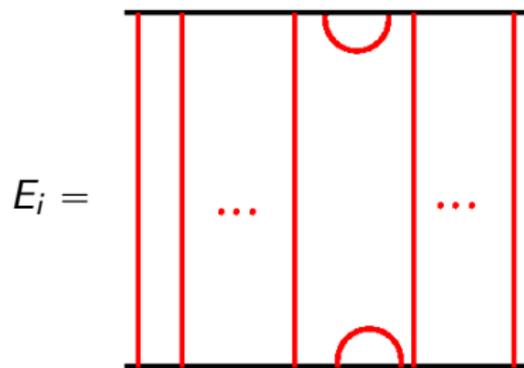
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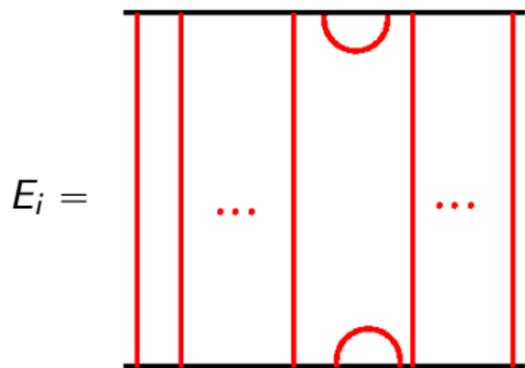
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Checking the e_i relations is then a very pleasant exercise. (The τ factor appears in a different place!)

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This has all been vastly generalised, but it should be stressed that the representation of the e'_j 's in the tensor power was in fact discovered in the subfactor context, by Pimsner and Popa!

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The e_i relations come from the nearest neighbour interaction between the spins. (David Evans.)

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Subfactors allow one to construct many other "anyonic" spin chains with other potential Hamiltonians.

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But the one that is of most interest to us is causality which asserts that, if \mathcal{O} and \mathcal{O}' are regions so that light cannot get from \mathcal{O} to \mathcal{O}' then

$$\mathcal{A}(\mathcal{O}) \text{ commutes with } \mathcal{A}(\mathcal{O}')$$

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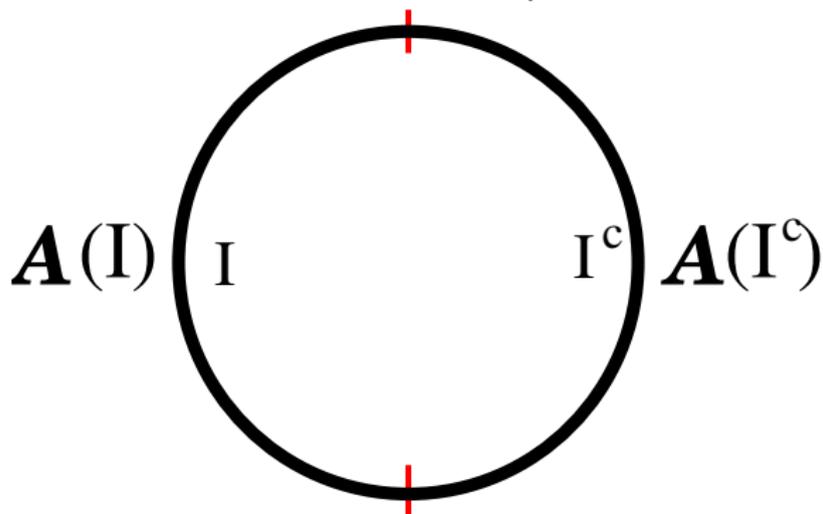
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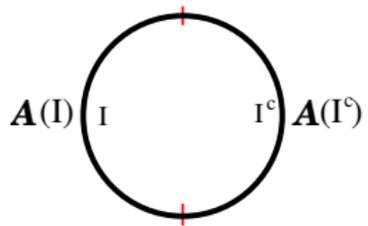
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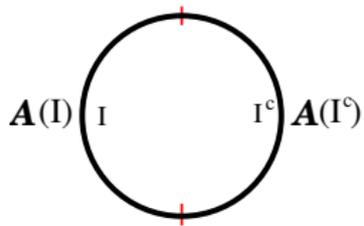
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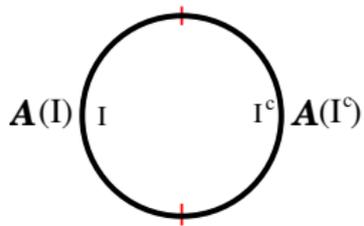
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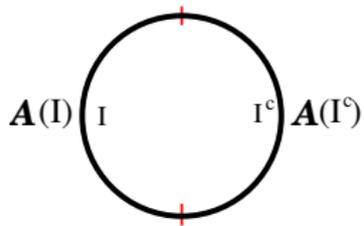


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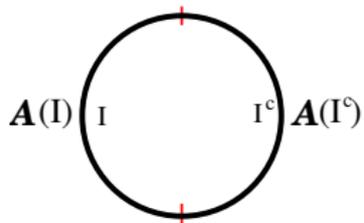
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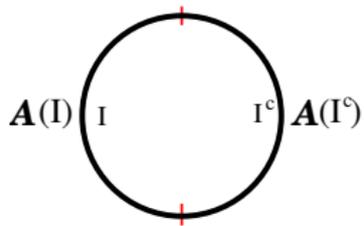
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This may take some time—perhaps even a good chunk of the 21st century.