Von Neumann Algebras meet Quantum Information Theory

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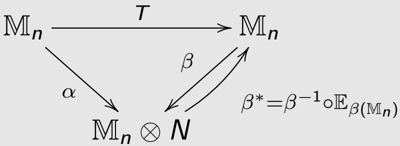
Definition (Delaroche '05): *Let $n \geq 2$. A unital quantum channel $T\colon \mathbb{M}_n \to \mathbb{M}_n$ is called* factorizable *if $\exists$ vN algebra N with normal faithful tracial state τ_N and unital $*$-homomorphisms (embeddings) $\alpha, \beta\colon \mathbb{M}_n \to \mathbb{M}_n \otimes N$ s.t. $T = \beta^* \circ \alpha$.*



We say T exactly factors through $\mathbb{M}_n \otimes N$, and N is the ancilla.

Note: N can be taken a II_1-vN alg (even a II_1-factor).

Theorem (Haagerup-M '11): T is factorizable iff $\exists$ vN algebra N with n.f. tracial state τ_N and $u \in \mathcal{U}(\mathbb{M}_n \otimes N)$ s.t.

$$Tx = (\mathrm{id}_{\mathbb{M}_n} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in \mathbb{M}_n.$$

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Example: $T \in \mathrm{Aut}(\mathbb{M}_n)$ exactly factors through $\mathbb{M}_n \otimes \mathbb{C}$.

The set $\mathcal{FM}(n)$ of factorizable unital quantum channels in dim n is convex and closed, and

$$\mathrm{conv}(\mathrm{Aut}(\mathbb{M}_n)) \subsetneq \mathcal{FM}(n), \quad \forall n \geq 3\,.$$

- $T \in \mathrm{conv}(\mathrm{Aut}(\mathbb{M}_n))$ iff T admits a finite dim abelian ancilla.

- Warning: The ancilla and its size are not uniquely determined!

E.g., if S_n is the *completely depolarizing channel* in dim $n \geq 2$,

$$S_n(x) = \mathrm{tr}_n(x)1_n, \quad x \in \mathbb{M}_n,$$

then both $\mathbb{C}^{n^2}$ and $\mathbb{M}_n$ are possible ancillas. Turns out that S_n also exactly factors through $(\mathbb{M}_n, \mathrm{tr}_n) * (\mathbb{M}_n, \mathrm{tr}_n)$!

Question (Delaroche): Are all quantum channels factorizable?

Proposition (Haagerup-M '11): Let $T : \mathbb{M}_n \to \mathbb{M}_n$ be a unital quantum channel ($n \geq 3$), with Choi canonical form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i, \quad x \in \mathbb{M}_n.$$

If $d \geq 2$ and $\{a_i^* a_j : 1 \leq i, j \leq d\}$ lin indep, then T not factoriz.

Example: With $a_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $a_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$,

$a_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ got first example of non-factorizable map.

(Holevo-Werner channel W_3^-)

(**Haagerup-M '11**): Non-factorizable maps are counterexamples to the Asymptotic Quantum Birkhoff Conjecture.

Question: Do we need vN algebras to describe factorizable maps?

Connections to the *Connes Embedding Problem* (CEP) whether every II_1-factor (on a sep Hilbert space) embeds in an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1-factor $\mathcal{R}$.

Let $\mathcal{FM}_{\text{matrix}}(n)$ and $\mathcal{FM}_{\text{fin}}(n)$ be the factorizable maps in dim $n \geq 3$ that admit a full matrix algebra as an ancilla, respectively, admit a finite dimensional C^*-algebra as an ancilla.

Theorem (Haagerup-M '15): TFAE

1. CEP has a positive answer.
2. $\forall n \geq 3 \; \forall T \in \mathcal{FM}(n) \; \exists$ an ancilla $(N, \tau_N) \hookrightarrow (\mathcal{R}^\omega, \tau_{\mathcal{R}^\omega})$.
3. $\forall n \geq 3 \; \forall T \in \mathcal{FM}(n) \; \exists (T_k)_{k \geq 1} \subset \mathcal{FM}_{\text{matrix}}(n)$ s.t. $\lim_{k \to \infty} \|T - T_k\|_{\text{cb}} = 0$.

(M '18): $\mathcal{FM}_{\text{fin}}(n) = \text{conv}(\mathcal{FM}_{\text{matrix}}(n))$, and $\mathcal{FM}_{\text{matrix}}(n)$ is non-convex and non-closed, whenever $n \geq 3$.

Theorem (M-Rørdam '18): $\mathcal{FM}_{\rm fin}(n)$ is not closed, $\forall n \geq 11$. Moreover, in each such dimens, there exist factorizable quantum channels that do require a type II_1 vN algebra as an ancilla.

The proof is based on analysis of sets of matrices of correlations arising from unitaries/projections in vN algebras (resp., finite dim C^*-algebras), and their closure properties.

$$\mathcal{F}_{\rm matr}(n) = \bigcup_{k\geq 1} \Big\{ \big[\mathrm{tr}_k(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in } \mathbb{M}_k \Big\},$$

$$\mathcal{F}_{\rm fin}(n) = \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite dim C}^*\text{-alg } (\mathcal{A}, \tau) \Big\},$$

$$\mathcal{G}(n) = \Big\{ \big[\tau(u_j^* u_i)\big] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (M, \tau) \Big\}.$$

Note: $\mathcal{F}_{\rm matr}(n) \subseteq \mathcal{F}_{\rm fin}(n) \subseteq \mathcal{G}(n)$. (All sets equal if $n = 2$.)

(**Kirchberg '93**): CEP positive iff $\mathcal{G}(n) = \mathrm{cl}(\mathcal{F}_{\rm matr}(n))$, $\forall n \geq 3$.

(**M-Rørdam '18**): $\mathcal{F}_{\rm matr}(n)$ is neither convex, nor closed $\forall n \geq 3$. Also, $\mathcal{F}_{\rm fin}(n)$ is not closed, $\forall n \geq 11$.

- (**Haagerup-M '11**): If $B \in \mathbb{M}_n$ is a *correlation* matrix, then its associated Schur multiplier T_B is factorizable iff $B \in \mathcal{G}(n)$. Furthermore, $T_B \in \mathcal{FM}_{\rm fin}(n)$ iff $B \in \mathcal{F}_{\rm fin}(n)$.

For $n \geq 2$, consider now the following sets of $n \times n$ matrices of correlations arising from projections:

$$\mathcal{D}(n) = \Big\{ \big[\tau(p_j p_i)\big] : p_1, \ldots, p_n \text{ projections in arbitrary } (M, \tau) \text{ finite vN alg} \Big\},$$

$$\mathcal{D}_{\mathrm{fin}}(n) = \Big\{ \big[\tau(p_j p_i)\big] : p_1, \ldots, p_n \text{ projections in arbitrary } (\mathcal{A}, \tau) \text{ finite dim C}^*\text{-alg} \Big\}.$$

- For $n \geq 2$, $\mathcal{D}(n)$ is closed and convex, and $\mathcal{D}_{\mathrm{fin}}(n)$ is convex. Also, $\mathcal{D}_{\mathrm{fin}}(2) = \mathcal{D}(2)$. Not known if $\mathcal{D}_{\mathrm{fin}}(3)$, $\mathcal{D}_{\mathrm{fin}}(4)$ are closed.

Note: CEP has positive answer iff $\mathcal{D}(n) = \mathrm{cl}(\mathcal{D}_{\mathrm{fin}}(n)),\ \forall n \geq 3$.

Theorem (M–Rørdam '18): $\mathcal{D}_{\mathrm{fin}}(n)$ not closed, $\forall n \geq 5$.

The proof follows ideas from **Dykema-Paulsen-Prakash '17**, but avoids graph correlation functions (and quantum games).

Projections adding up to a scalar multiple of the identity operator:

Let Σ_n be the set of $\alpha \geq 0$ for which $\exists$ projections $p_1, \dots, p_n$ on a Hilbert space H such that $\sum_{j=1}^n p_j = \alpha \cdot I_H$.

- It is known that $\Sigma_n \subset \mathbb{Q}$, when $n \leq 4$.

Theorem (**Kruglyak-Rabanovich-Samoilenko '02**): Let $n \geq 5$. There exist projections $p_1, \dots, p_n$ on a *finite dimensional* Hilbert space H so that $\sum_{j=1}^n p_j = \alpha \cdot I_H$ if and only if $\alpha \in \Sigma_n \cap \mathbb{Q}$. Furthermore,

$$\left[\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})\right] \subseteq \Sigma_n.$$

Note: The "only if" part is easy (with Tr standard trace on $B(H)$):

$$\sum_{j=1}^n p_j = \alpha \cdot I_H \implies \alpha \cdot \dim(H) = \sum_{j=1}^n \mathrm{Tr}(p_j).$$

For $n \geq 2$ and $1/n \leq t \leq 1$, consider the following $n \times n$ matrix:

$$A_t^{(n)}(i,j) = \begin{cases} t, & i = j, \\ \dfrac{t(nt-1)}{n-1}, & i \neq j. \end{cases}$$

Proposition: Let $(\mathcal{A}, \tau)$ be a unital C^*-alg with faithful tracial state τ, and $p_1, \ldots, p_n \in \mathcal{A}$ be projections. Set $\alpha = nt$.

- If

$$\tau(p_j p_i) = A_t^{(n)}(i,j), \qquad 1 \leq i,j \leq n,$$

then $\sum_{j=1}^{n} p_j = \alpha \cdot 1_{\mathcal{A}}$. Moreover, if $t \notin \mathbb{Q}$, then $\dim(\mathcal{A}) = \infty$. (Even stronger, $\mathcal{A}$ has no finite dimens repres.)

- Respectively, if $\sum_{j=1}^{n} p_j = \alpha \cdot 1_{\mathcal{A}}$, then $\exists m \geq 1$ and projections $\widetilde{p}_1, \ldots, \widetilde{p}_n \in M_m(\mathcal{A})$ such that

$$(\tau \otimes \mathrm{tr_m})(\widetilde{p}_j \widetilde{p}_i) = A_t^{(n)}(i,j), \qquad 1 \leq i,j \leq n.$$

Recall

$$A_t^{(n)}(i,j) = \begin{cases} t, & i = j, \\ \dfrac{t(nt-1)}{n-1}, & i \neq j. \end{cases}$$

Combining the previous proposition with the theorem of Kruglyak, Rabanovich and Samoilenko, we get

Theorem: Let $n \geq 5$, $t \in [\frac{1}{2}(1-\sqrt{1-4/n}), \frac{1}{2}(1+\sqrt{1-4/n})]$.

- If $t \in \mathbb{Q}$, then $A_t^{(n)} \in \mathcal{D}_{\mathrm{fin}}(n)$.
- If $t \notin \mathbb{Q}$, then $A_t^{(n)} \in \mathrm{cl}(\mathcal{D}_{\mathrm{fin}}(n)) \setminus \mathcal{D}_{\mathrm{fin}}(n)$.

In particular, $\mathcal{D}_{\mathrm{fin}}(n)$ is non-closed, when $n \geq 5$.

Note: If $t \in \frac{1}{n}\Sigma_n \setminus \mathbb{Q}$, and $p_1, \ldots, p_n$ proj in a finite vN alg (N, τ_N) s.t. $\tau(p_j p_i) = A_t^{(n)}(i,j)$, $1 \leq i,j \leq n$, then N must be of type II_1.

Theorem (M-Rørdam): $\mathcal{D}_{\text{fin}}(n)$ is not closed, for all $n \geq 5$.

Using a trick originating in ideas of **Regev-Slofstra-Vidick**, we can prove that

$$\mathcal{D}_{\text{fin}}(n) \text{ not closed} \Longrightarrow \mathcal{F}_{\text{fin}}(2n+1) \text{ not closed.}$$

We conclude that $\mathcal{F}_{\text{fin}}(n)$ is not closed, $\forall n \geq 11$.

The trick (originating in ideas of **Regev-Slofstra-Vidick**):
Let $p_1, \dots, p_n \in M$ be projections in a finite vN alg M with n.f. tracial state τ_M. Define unitaries $u_0, u_1, \dots u_{2n} \in M$ by $u_0 = 1$ and

$$u_j = 2p_j - 1, \; 1 \le j \le n, \quad u_j = \frac{1}{\sqrt{2}}(u_{j-n} + i \cdot 1), \; n+1 \le j \le 2n.$$

Let (N, τ_N) be another finite vN alg with n.f. tracial state. Then $\exists$ unitaries $v_0, v_1, \dots, v_{2n} \in N$ satisfying

$$\tau_N(v_j^* v_i) = \tau_M(u_j^* u_i), \qquad 0 \le i, j \le 2n, \qquad (*)$$

iff $\exists$ projections $q_1, \dots, q_n \in N$ satisfying

$$\tau_N(q_j q_i) = \tau_M(p_j p_i), \qquad 1 \le i, j \le n. \qquad (**)$$