Von Neumann Algebras meet Quantum Information Theory

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Definition (Delaroche ‘05): Let \( n \geq 2 \). A unital quantum channel \( T : \mathbb{M}_n \to \mathbb{M}_n \) is called factorizable if \( \exists \) vN algebra \( N \) with normal faithful tracial state \( \tau_N \) and unital \( * \)-homomorphisms (embeddings) \( \alpha, \beta : \mathbb{M}_n \to \mathbb{M}_n \otimes N \) s.t. \( T = \beta^* \circ \alpha \).

\[
\begin{array}{ccc}
\mathbb{M}_n & \xrightarrow{T} & \mathbb{M}_n \\
\downarrow{\alpha} & & \downarrow{\beta} \\
\mathbb{M}_n \otimes N & \xrightarrow{\beta^* = \beta^{-1} \circ E_\beta(\mathbb{M}_n)} & \\
\end{array}
\]

We say \( T \) exactly factors through \( \mathbb{M}_n \otimes N \), and \( N \) is the ancilla.

Note: \( N \) can be taken a II\(_1\)-vN alg (even a II\(_1\)-factor).

Theorem (Haagerup-M ‘11): \( T \) is factorizable iff \( \exists \) vN algebra \( N \) with n.f. tracial state \( \tau_N \) and \( u \in \mathcal{U}(\mathbb{M}_n \otimes N) \) s.t.

\[
Tx = (\text{id}_{\mathbb{M}_n} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in \mathbb{M}_n.
\]
Theorem (Haagerup-M ‘11): \( T \) is factorizable iff \( \exists \) vN algebra \( N \) with n.f. tracial state \( \tau_N \) and \( u \in \mathcal{U}(\mathbb{M}_n \otimes N) \) s.t.
\[
T x = (\text{id}_{\mathbb{M}_n} \otimes \tau_N)(u^*(x \otimes 1_N)u), \quad x \in \mathbb{M}_n.
\]

Example: \( T \in \text{Aut}(\mathbb{M}_n) \) exactly factors through \( \mathbb{M}_n \otimes \mathbb{C} \).

The set \( \mathcal{FM}(n) \) of factorizable unital quantum channels in dim \( n \) is convex and closed, and

\[\text{conv} (\text{Aut} (\mathbb{M}_n)) \subsetneq \mathcal{FM}(n), \quad \forall n \geq 3.\]

\[\Rightarrow \quad T \in \text{conv} (\text{Aut} (\mathbb{M}_n)) \text{ iff } T \text{ admits a finite dim abelian ancilla.}\]

\[\Rightarrow \text{ Warning: The ancilla and its size are not uniquely determined!}\]
E.g., if \( S_n \) is the \textit{completely depolarizing channel} in dim \( n \geq 2 \),
\[
S_n(x) = \text{tr}_n(x)1_n, \quad x \in \mathbb{M}_n,
\]
then both \( \mathbb{C}^{n^2} \) and \( \mathbb{M}_n \) are possible ancillas. Turns out that \( S_n \) also
exactly factors through \((\mathbb{M}_n, \text{tr}_n) \ast (\mathbb{M}_n, \text{tr}_n)\)!
**Question (Delaroche):** Are all quantum channels factorizable?

**Proposition (Haagerup-M ’11):** Let $T : \mathbb{M}_n \to \mathbb{M}_n$ be a unital quantum channel ($n \geq 3$), with Choi canonical form

$$Tx = \sum_{i=1}^{d} a_i^* x a_i, \quad x \in \mathbb{M}_n.$$ 

If $d \geq 2$ and $\{a_i^* a_j : 1 \leq i, j \leq d\}$ lin indep, then $T$ not factoriz.

**Example:** With $a_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$, $a_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, $a_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ got first example of non-factorizable map. 

(Holevo-Werner channel $W_3^-$)

(Haagerup-M ‘11): Non-factorizable maps are counterexamples to the Asymptotic Quantum Birkhoff Conjecture.)
Question: Do we need vN algebras to describe factorizable maps?

Connections to the *Connes Embedding Problem (CEP)* whether every II$_1$-factor (on a sep Hilbert space) embeds in an ultrapower $\mathcal{R}^\omega$ of the hyperfinite II$_1$-factor $\mathcal{R}$.

Let $\mathcal{F}M_{\text{matrix}}(n)$ and $\mathcal{F}M_{\text{fin}}(n)$ be the factorizable maps in dim $n \geq 3$ that admit a full matrix algebra as an ancilla, respectively, admit a finite dimensional $C^*$-algebra as an ancilla.

**Theorem (Haagerup-M ’15):** TFAE

1. CEP has a positive answer.
2. $\forall n \geq 3 \ \forall T \in \mathcal{F}M(n) \ \exists$ an ancilla $(N, \tau_N) \hookrightarrow (\mathcal{R}^\omega, \tau_{\mathcal{R}^\omega})$.
3. $\forall n \geq 3 \ \forall T \in \mathcal{F}M(n) \ \exists (T_k)_{k \geq 1} \subset \mathcal{F}M_{\text{matrix}}(n)$ s.t.
   \[ \lim_{k \to \infty} \| T - T_k \|_{\text{cb}} = 0. \]

(M ‘18): $\mathcal{F}M_{\text{fin}}(n) = \text{conv}(\mathcal{F}M_{\text{matrix}}(n))$, and $\mathcal{F}M_{\text{matrix}}(n)$ is non-convex and non-closed, whenever $n \geq 3$. 

**Theorem (M-Rørdam ‘18):** $\mathcal{FM}_{\text{fin}}(n)$ is not closed, $\forall n \geq 11$. Moreover, in each such dimens, there exist factorizable quantum channels that do require a type $\text{II}_1$ vN algebra as an ancilla.

The proof is based on analysis of sets of matrices of correlations arising from unitaries/projections in vN algebras (resp., finite dim C*-algebras), and their closure properties.
\[ \mathcal{F}_{\text{matr}}(n) = \bigcup_{k \geq 1} \left\{ \left[ \text{tr}_k(u^*_j u_i) \right] : u_1, \ldots, u_n \text{ unitaries in } \mathbb{M}_k \right\}, \]
\[ \mathcal{F}_{\text{fin}}(n) = \left\{ \left[ \tau(u^*_j u_i) \right] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite dim } \mathbb{C}^*-\text{alg } (\mathcal{A}, \tau) \right\}, \]
\[ \mathcal{G}(n) = \left\{ \left[ \tau(u^*_j u_i) \right] : u_1, \ldots, u_n \text{ unitaries in arbitrary finite vN alg } (\mathcal{M}, \tau) \right\}. \]

**Note:** \( \mathcal{F}_{\text{matr}}(n) \subseteq \mathcal{F}_{\text{fin}}(n) \subseteq \mathcal{G}(n). \) (All sets equal if \( n = 2. \))

(Kirchberg ‘93): CEP positive iff \( \mathcal{G}(n) = \text{cl}(\mathcal{F}_{\text{matr}}(n)), \forall n \geq 3. \)

(M-Rørdam ‘18): \( \mathcal{F}_{\text{matr}}(n) \) is neither convex, nor closed \( \forall n \geq 3. \)
Also, \( \mathcal{F}_{\text{fin}}(n) \) is not closed, \( \forall n \geq 11. \)

▶ (Haagerup-M ‘11): If \( B \in \mathbb{M}_n \) is a correlation matrix, then its associated Schur multiplier \( T_B \) is factorizable iff \( B \in \mathcal{G}(n). \)
Furthermore, \( T_B \in \mathcal{F}_\mathbb{M}_{\text{fin}}(n) \) iff \( B \in \mathcal{F}_{\text{fin}}(n). \)
For \( n \geq 2 \), consider now the following sets of \( n \times n \) matrices of correlations arising from projections:

\[
\mathcal{D}(n) = \left\{ [\tau(p_j p_i)] : p_1, \ldots, p_n \text{ projections in arbitrary } (M, \tau) \text{ finite vN alg} \right\},
\]

\[
\mathcal{D}_{\text{fin}}(n) = \left\{ [\tau(p_j p_i)] : p_1, \ldots, p_n \text{ projections in arbitrary } (\mathcal{A}, \tau) \text{ finite dim } \mathbb{C}^*\text{-alg} \right\}.
\]

\( \blacktriangleright \) For \( n \geq 2 \), \( \mathcal{D}(n) \) is closed and convex, and \( \mathcal{D}_{\text{fin}}(n) \) is convex. Also, \( \mathcal{D}_{\text{fin}}(2) = \mathcal{D}(2) \). Not known if \( \mathcal{D}_{\text{fin}}(3), \mathcal{D}_{\text{fin}}(4) \) are closed.

Note: CEP has positive answer iff \( \mathcal{D}(n) = \text{cl}(\mathcal{D}_{\text{fin}}(n)), \forall n \geq 3 \).

**Theorem (M–Rørdam ‘18):** \( \mathcal{D}_{\text{fin}}(n) \) not closed, \( \forall n \geq 5 \).

The proof follows ideas from Dykema-Paulsen-Prakash ‘17, but avoids graph correlation functions (and quantum games).
Projections adding up to a scalar multiple of the identity operator:

Let $\Sigma_n$ be the set of $\alpha \geq 0$ for which $\exists$ projections $p_1, \ldots, p_n$ on a Hilbert space $H$ such that $\sum_{j=1}^{n} p_j = \alpha \cdot I_H$.

It is known that $\Sigma_n \subset \mathbb{Q}$, when $n \leq 4$.

**Theorem (Kruglyak-Rabanovich-Samoilenko ‘02):** Let $n \geq 5$. There exist projections $p_1, \ldots, p_n$ on a *finite dimensional* Hilbert space $H$ so that $\sum_{j=1}^{n} p_j = \alpha \cdot I_H$ if and only if $\alpha \in \Sigma_n \cap \mathbb{Q}$.

Furthermore,

$$\left[\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})\right] \subseteq \Sigma_n.$$  

**Note:** The “only if” part is easy (with $\text{Tr}$ standard trace on $B(H)$):

$$\sum_{j=1}^{n} p_j = \alpha \cdot I_H \implies \alpha \cdot \dim(H) = \sum_{j=1}^{n} \text{Tr}(p_j).$$
For $n \geq 2$ and $1/n \leq t \leq 1$, consider the following $n \times n$ matrix:

$$A_t^{(n)}(i, j) = \begin{cases} t, & i = j, \\ \frac{t(nt - 1)}{n-1}, & i \neq j. \end{cases}$$

**Proposition:** Let $(\mathcal{A}, \tau)$ be a unital $C^*$-alg with faithful tracial state $\tau$, and $p_1, \ldots, p_n \in \mathcal{A}$ be projections. Set $\alpha = n t$.

- If
  $$\tau(p_j p_i) = A_t^{(n)}(i, j), \quad 1 \leq i, j \leq n,$$
  then $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$. Moreover, if $t \notin \mathbb{Q}$, then $\dim(\mathcal{A}) = \infty$. (Even stronger, $\mathcal{A}$ has no finite dimens repres.)

- Respectively, if $\sum_{j=1}^n p_j = \alpha \cdot 1_{\mathcal{A}}$, then $\exists m \geq 1$ and projections $\tilde{p}_1, \ldots, \tilde{p}_n \in M_m(\mathcal{A})$ such that
  $$(\tau \otimes \text{tr}_m)(\tilde{p}_j \tilde{p}_i) = A_t^{(n)}(i, j), \quad 1 \leq i, j \leq n.$$
Recall

\[ A_t^{(n)}(i,j) = \begin{cases} 
  t, & i = j, \\
  \frac{t(nt - 1)}{n - 1}, & i \neq j.
\end{cases} \]

Combining the previous proposition with the theorem of Kruglyak, Rabanovich and Samoilenko, we get

**Theorem:** Let \( n \geq 5 \), \( t \in \left[ \frac{1}{2}(1 - \sqrt{1 - 4/n}), \frac{1}{2}(1 + \sqrt{1 - 4/n}) \right] \).

- If \( t \in \mathbb{Q} \), then \( A_t^{(n)} \in \mathcal{D}_{\text{fin}}(n) \).
- If \( t \notin \mathbb{Q} \), then \( A_t^{(n)} \in \text{cl}(\mathcal{D}_{\text{fin}}(n)) \setminus \mathcal{D}_{\text{fin}}(n) \).

In particular, \( \mathcal{D}_{\text{fin}}(n) \) is non-closed, when \( n \geq 5 \).

**Note:** If \( t \in \frac{1}{n} \Sigma_n \setminus \mathbb{Q} \), and \( p_1, \ldots, p_n \) proj in a finite vN alg \( (N, \tau_N) \) s.t. \( \tau(p_j p_i) = A_t^{(n)}(i,j), 1 \leq i, j \leq n \), then \( N \) must be of type II\(_1\).
Theorem (M-Rørdam): $D_{\text{fin}}(n)$ is not closed, for all $n \geq 5$.

Using a trick originating in ideas of Regev-Slofstra-Vidick, we can prove that

$D_{\text{fin}}(n)$ not closed $\implies F_{\text{fin}}(2n + 1)$ not closed.

We conclude that $F_{\text{fin}}(n)$ is not closed, $\forall n \geq 11$. 

The trick (originating in ideas of Regev-Slofstra-Vidick):
Let \( p_1, \ldots, p_n \in M \) be projections in a finite vN alg \( M \) with n.f. tracial state \( \tau_M \). Define unitaries \( u_0, u_1, \ldots, u_{2n} \in M \) by \( u_0 = 1 \) and
\[
\begin{align*}
    u_j &= 2p_j - 1, \quad 1 \leq j \leq n, \\
    u_j &= \frac{1}{\sqrt{2}}(u_{j-n} + i \cdot 1), \quad n + 1 \leq j \leq 2n.
\end{align*}
\]

Let \((N, \tau_N)\) be another finite vN alg with n.f. tracial state. Then \( \exists \) unitaries \( v_0, v_1, \ldots, v_{2n} \in N \) satisfying
\[
\tau_N(v_j^* v_i) = \tau_M(u_j^* u_i), \quad 0 \leq i, j \leq 2n, \quad (*)
\]
iff \( \exists \) projections \( q_1, \ldots, q_n \in N \) satisfying
\[
\tau_N(q_j q_i) = \tau_M(p_j p_i), \quad 1 \leq i, j \leq n. \quad (**)
\]