The Pure Extension Property for Discrete Crossed Products

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Remembering Dick Kadison

(Ed & Rita Effros with Dick Kadison, Philadelphia, 1967)

The Pure Extension Property

Definition (C^*-inclusion)

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Definition (pure extension property)

We say that a C^*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ has the **pure extension property (PEP)** if every pure state on $\mathcal{A}$ extends uniquely to a (pure) state on $\mathcal{B}$.

Kadison-Singer 1959

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Proof.

There are multiple conditional expectations $B(L^2) \to L^\infty$, all of which are singular. □

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Theorem (Marcus-Spielman-Srivastava 2013)

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PEP recommended reading

- Kadison and Singer 1959
- Anderson 1979
- Archbold, Bunce (J.), and Gregson 1982
- Batty 1982
- Tomiyama 1987
- Bunce (L.) and Chu 1998
- Archbold 1999
- Renault 2008
- Akemann, Wassermann (S.), and Weaver 2010
- Akemann and Sherman 2012
- Popa 2014
- Marcus, Spielman, and Srivastava 2015
- Popa and Vaes 2015

Topological Dynamical Systems

Let $G \curvearrowright X$ be the action of a discrete group on a topological space by homeomorphisms. We denote by

$$\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$$

the **fixed points** of $g \in G$, and by

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Definition (free topological dynamical system)

We say that $G \curvearrowright X$ is **free** if one of the following equivalent conditions holds:

1. $g \cdot x \neq x$ for all $e \neq g \in G$ and all $x \in X$;
2. $\mathrm{Fix}(g) = \emptyset$ for all $e \neq g \in G$;
3. $G_x = \{e\}$ for all $x \in X$.

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Theorem (Batty 1982; Tomiyama 1987)

The C^*-inclusion $C(X) \subseteq C(X) \rtimes_r G$ has the PEP if and only if $G \curvearrowright X$ is free.

C^*-Dynamical Systems

Proposition

Let $(\mathcal{A}, G, \alpha)$ be a discrete C^*-dynamical system. Then there is a corresponding topological dynamical system $G \curvearrowright \widehat{\mathcal{A}}$ given by

$$g \cdot [\pi] = [\pi \circ \alpha_g^{-1}].$$

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Theorem (the main result)

The C^*-inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$ has the PEP if and only if $G \curvearrowright \widehat{\mathcal{A}}$ is free.

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Theorem (the main result, localized)

For a pure state $\phi \in PS(\mathcal{A})$, the following are equivalent:

1. *ϕ extends uniquely to a pure state on $\mathcal{A} \rtimes_{\alpha,r} G$;*
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1. *ϕ extends uniquely to a pure state on $\mathcal{A} \rtimes_{\alpha,r} G$;*
2. *$G_{[\pi_\phi]} = \{e\}$;*
3. *$\pi_\phi : \mathcal{A} \to B(\mathcal{H}_\phi)$ extends uniquely to a UCP map $\theta : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H}_\phi)$.*

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For a pure state $\phi \in PS(\mathcal{A})$, the following are equivalent:

1. ϕ *extends uniquely to a pure state on* $\mathcal{A} \rtimes_{\alpha,r} G$;
2. $G_{[\pi_\phi]} = \{e\}$;
3. $\pi_\phi : \mathcal{A} \to B(\mathcal{H}_\phi)$ *extends uniquely to a UCP map* $\theta : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H}_\phi)$.

Theorem (the main result, localized even more)

For a non-zero irreducible representation $\pi : \mathcal{A} \to B(\mathcal{H})$, the following are equivalent:

1. $g \cdot [\pi] \neq [\pi]$
2. $\theta(g) = 0$ *for every UCP map* $\theta : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H})$ *extending* π.

$\neg 2 \implies \neg 1$

Proof.

Suppose that $\theta(g) \neq 0$, where $\theta : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H})$ is a UCP map extending the non-zero irreducible representation $\pi : \mathcal{A} \to B(\mathcal{H})$. For all $a \in \mathcal{A}$,

$$\pi(a)\theta(g) = \theta(ag) = \theta(g\alpha_g^{-1}(a)) = \theta(g)\pi(\alpha_g^{-1}(a)).$$

By an argument of Choda-Kasahara-Nakamoto,

$$\theta(g)^*\theta(g) = \theta(g)\theta(g)^* \in \pi(\mathcal{A})' = \mathbb{C}\, I.$$

Thus $\theta(g) = tU$ for some unitary $U \in B(\mathcal{H})$ and some $t > 0$, which implies

$$\pi(\alpha_g^{-1}(a)) = U^*\pi(a)U, \; a \in \mathcal{A}.$$

Therefore

$$g \cdot [\pi] = [\pi].$$

□

$\neg 1 \implies \neg 2$

Proof.

Suppose $g \cdot [\pi] = [\pi]$. Then there exists a unitary $U \in B(\mathcal{H})$ such that

$$\pi(\alpha_g(a)) = U\pi(a)U^*, \ a \in \mathcal{A}.$$

Define a CB map $\theta : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H})$ by

$$\theta(x) = \pi(\mathbb{E}(xg^{-1}))U, \ x \in \mathcal{A} \rtimes_{\alpha,r} G.$$

Note that $\theta(g) = U \neq 0$. Also note that θ is $\mathcal{A}$-bimodular with respect to π, meaning

$$\theta(ax) = \pi(a)\theta(x) \text{ and } \theta(xa) = \theta(x)\pi(a), \ a \in \mathcal{A}, \ x \in \mathcal{A} \rtimes_{\alpha,r} G.$$

By a variant of a result of Wittstock,

$$\theta = (\theta_1 - \theta_2) + i(\theta_3 - \theta_4),$$

where for each $1 \leq j \leq 4$, $\theta_j : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H})$ is a CP map which is $\mathcal{A}$-bimodular with respect to π. Without loss of generality, $\theta_1(g) \neq 0$. By a variant of a result of Effros-Ruan,

$$\theta_1(x) = \theta_1(1)^{1/2}\tilde{\theta}_1(x)\theta_1(1)^{1/2}, \ x \in \mathcal{A} \rtimes_{\alpha,r} G,$$

where $\tilde{\theta}_1 : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H})$ is a UCP map which is $\mathcal{A}$-bimodular with respect to π. Clearly $\tilde{\theta}_1(g) \neq 0$. Thus $\tilde{\theta}_1$ and $\pi \circ \mathbb{E}$ are distinct UCP extensions of π. □

Unique Extension Properties for Discrete Crossed Products

Theorem

The C^*-inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$

- *has the* **pure extension property** $\iff$ $G \curvearrowright \widehat{\mathcal{A}}$ *is* **free**;
- *has the* **almost extension property** *of Nagy and Reznikoff* $\iff$ $G \curvearrowright \widehat{\mathcal{A}}$ *is* **essentially free**;
- *has a* **unique pseudo-expectation** *in the sense of Pitts* $\iff$ $G \curvearrowright \mathcal{A}$ *is* **properly outer** *in the sense of Kishimoto;*
- *has a* **unique conditional expectation** $\iff$ $G \curvearrowright \mathcal{A}$ *is* **freely acting**.

Examples

Example

The C^*-inclusion $C(\mathbb{T}) \subseteq C(\mathbb{T}) \rtimes_\theta \mathbb{Z}$ (irrational rotation) has the PEP.

Example

The C^*-inclusion $\mathcal{O}_2 \subseteq \mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ (switch the generators) has the AEP but not the PEP.

Example

The C^*-inclusion $C(\mathbb{T}) \subseteq C(\mathbb{T}) \rtimes O(2)$ has a unique pseudo-expectation but not the AEP.

Example

The C^*-inclusion $K(\ell^2)^1 \subseteq K(\ell^2)^1 \rtimes_\alpha \mathbb{Z}_2$ where $\alpha = \mathrm{Ad}(U)$ and $U = \bigoplus_{n=1}^\infty \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in B(\ell^2)$, has a unique conditional expectation but infinitely many pseudo-expectations parameterized by the interval $[-1, 1]$.

References

Zarikian, Vrej *The Pure Extension Property for Discrete Crossed Products*, Houston Journal of Mathematics 45 (2019), 233-243.

Zarikian, Vrej *Unique Expectations for Discrete Crossed Products*, Annals of Functional Analysis 10 (2019), 60-71.

Thanks

Thanks!

Questions?

Personal Dick Kadison stories

Aug. 2000 MSRI (Clay Mathematics Institute Introductory Workshop in Operator Algebras)
May 2003 GPOTS Illinois
Oct. 2004 ECOAS USNA
Mar. 2011 Penn Analysis Seminar - "Toeplitz CAR Flows" by Izumi