The Pure Extension Property for Discrete Crossed Products

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Operator Algebras in the 21st Century
A Conference in Memory of Richard V. Kadison
March 30, 2019
Remembering Dick Kadison

(Ed & Rita Effros with Dick Kadison, Philadelphia, 1967)
Definition (C*-inclusion)

A C*-inclusion is an inclusion of unital C*-algebras $A \subseteq B$, such that $1_A = 1_B$. 

The Pure Extension Property
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Definition (pure extension property)

We say that a C*-inclusion $\mathcal{A} \subseteq \mathcal{B}$ has the pure extension property (PEP) if every pure state on $\mathcal{A}$ extends uniquely to a (pure) state on $\mathcal{B}$.
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The Pure Extension Property

Kadison-Singer 1959

Theorem (K-S 1959)

The $C^*$-inclusion $L^\infty \subseteq B(L^2)$ does not have the PEP.
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Proof.

There are multiple conditional expectations $B(L^2) \to L^\infty$, all of which are singular.
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*Does the $C^*$-inclusion $\ell^\infty \subseteq B(\ell^2)$ have the PEP?*
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Theorem (Marcus-Spielman-Srivastava 2013)

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The Pure Extension Property

PEP recommended reading

- Kadison and Singer 1959
- Anderson 1979
- Archbold, Bunce (J.), and Gregson 1982
- Batty 1982
- Tomiyama 1987
- Bunce (L.) and Chu 1998
- Archbold 1999
- Renault 2008
- Akemann, Wassermann (S.), and Weaver 2010
- Akemann and Sherman 2012
- Popa 2014
- Marcus, Spielman, and Srivastava 2015
- Popa and Vaes 2015
Topological Dynamical Systems

Let $G \curvearrowright X$ be the action of a discrete group on a topological space by homeomorphisms. We denote by

$$\text{Fix}(g) = \{x \in X : g \cdot x = x\}$$

the **fixed points** of $g \in G$, and by

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**Definition (free topological dynamical system)**

We say that $G \acts X$ is **free** if one of the following equivalent conditions holds:

1. $g \cdot x \neq x$ for all $e \neq g \in G$ and all $x \in X$;
2. $\text{Fix}(g) = \emptyset$ for all $e \neq g \in G$;
3. $G_x = \{e\}$ for all $x \in X$. 

Theorem (Batty 1982; Tomiyama 1987)

The $C^*$-inclusion $C(X) \subseteq C(X) \rtimes_r G$ has the PEP if and only if $G \acts X$ is free.
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**Theorem (Batty 1982; Tomiyama 1987)**

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C*-Dynamical Systems

Proposition

Let \((\mathcal{A}, G, \alpha)\) be a discrete C*-dynamical system. Then there is a corresponding topological dynamical system \(G \curvearrowright \hat{\mathcal{A}}\) given by

\[ g \cdot [\pi] = [\pi \circ \alpha_g^{-1}] \]
Proposition

Let $(\mathcal{A}, G, \alpha)$ be a discrete $C^*$-dynamical system. Then there is a corresponding topological dynamical system $G \rtimes \hat{\mathcal{A}}$ given by

$$g \cdot [\pi] = [\pi \circ \alpha_g^{-1}].$$

Theorem (the main result)

The $C^*$-inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$ has the PEP if and only if $G \rtimes \hat{\mathcal{A}}$ is free.
Localizations

Theorem (the main result)

*The $C^*$-inclusion $A \subseteq A \rtimes_{\alpha, r} G$ has the PEP if and only if $G \curvearrowright \hat{A}$ is free.*
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Localizations

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Theorem (the main result, localized)

For a pure state $\phi \in \text{PS}(\mathcal{A})$, the following are equivalent:

1. $\phi$ extends uniquely to a pure state on $\mathcal{A} \rtimes_{\alpha,r} G$;
2. $G[\pi\phi] = \{e\}$;
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1. $\phi$ extends uniquely to a pure state on $\mathcal{A} \rtimes_{\alpha, r} G$;
2. $G[\pi_\phi] = \{e\}$;
3. $\pi_\phi : \mathcal{A} \to B(\mathcal{H}_\phi)$ extends uniquely to a UCP map $\theta : \mathcal{A} \rtimes_{\alpha, r} G \to B(\mathcal{H}_\phi)$.
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Localizations

Theorem (the main result)

*The C*-inclusion* \( A \subseteq A \rtimes_{\alpha, r} G \) *has the PEP* if and only if \( G \curvearrowright \hat{A} \) is free.

Theorem (the main result, localized)

*For a pure state* \( \phi \in PS(A) \), the following are equivalent:

1. \( \phi \) extends uniquely to a pure state on \( A \rtimes_{\alpha, r} G \);
2. \( G[\pi_{\phi}] = \{e\} \);
3. \( \pi_{\phi} : A \to B(\mathcal{H}_{\phi}) \) extends uniquely to a UCP map \( \theta : A \rtimes_{\alpha, r} G \to B(\mathcal{H}_{\phi}) \).

Theorem (the main result, localized even more)

*For a non-zero irreducible representation* \( \pi : A \to B(\mathcal{H}) \), the following are equivalent:

1. \( g \cdot [\pi] \neq [\pi] \)
2. \( \theta(g) = 0 \) for every UCP map \( \theta : A \rtimes_{\alpha, r} G \to B(\mathcal{H}) \) extending \( \pi \).
Proof.

Suppose that $\theta(g) \neq 0$, where $\theta : \mathcal{A} \rtimes_{\alpha, r} G \to B(\mathcal{H})$ is a UCP map extending the non-zero irreducible representation $\pi : \mathcal{A} \to B(\mathcal{H})$. For all $a \in \mathcal{A}$,

$$\pi(a)\theta(g) = \theta(ag) = \theta(g\alpha_g^{-1}(a)) = \theta(g)\pi(\alpha_g^{-1}(a)).$$

By an argument of Choda-Kasahara-Nakamoto,

$$\theta(g)^*\theta(g) = \theta(g)\theta(g)^* \in \pi(\mathcal{A})' = \mathbb{C}I.$$

Thus $\theta(g) = tU$ for some unitary $U \in B(\mathcal{H})$ and some $t > 0$, which implies

$$\pi(\alpha_g^{-1}(a)) = U^*\pi(a)U, \ a \in \mathcal{A}.$$

Therefore

$$g \cdot [\pi] = [\pi].$$
Proof.

Suppose \( g \cdot [\pi] = [\pi] \). Then there exists a unitary \( U \in B(\mathcal{H}) \) such that

\[
\pi(\alpha_g(a)) = U\pi(a)U^*, \ a \in A.
\]

Define a CB map \( \theta : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H}) \) by

\[
\theta(x) = \pi(E(xg^{-1}))U, \ x \in \mathcal{A} \rtimes_{\alpha,r} G.
\]

Note that \( \theta(g) = U \neq 0 \). Also note that \( \theta \) is \( \mathcal{A} \)-bimodular with respect to \( \pi \), meaning

\[
\theta(ax) = \pi(a)\theta(x) \text{ and } \theta(xa) = \theta(x)\pi(a), \ a \in \mathcal{A}, \ x \in \mathcal{A} \rtimes_{\alpha,r} G.
\]

By a variant of a result of Wittstock,

\[
\theta = (\theta_1 - \theta_2) + i(\theta_3 - \theta_4),
\]

where for each \( 1 \leq j \leq 4 \), \( \theta_j : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H}) \) is a CP map which is \( \mathcal{A} \)-bimodular with respect to \( \pi \). Without loss of generality, \( \theta_1(g) \neq 0 \). By a variant of a result of Effros-Ruan,

\[
\theta_1(x) = \theta_1(1)^{1/2}\tilde{\theta}_1(x)\theta_1(1)^{1/2}, \ x \in \mathcal{A} \rtimes_{\alpha,r} G,
\]

where \( \tilde{\theta}_1 : \mathcal{A} \rtimes_{\alpha,r} G \to B(\mathcal{H}) \) is a UCP map which is \( \mathcal{A} \)-bimodular with respect to \( \pi \). Clearly \( \tilde{\theta}_1(g) \neq 0 \). Thus \( \tilde{\theta}_1 \) and \( \pi \circ E \) are distinct UCP extensions of \( \pi \).
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Unique Extension Properties for Discrete Crossed Products

Theorem

The $C^*$-inclusion $\mathcal{A} \subseteq \mathcal{A} \rtimes_{\alpha,r} G$

- has the **pure extension property** $\iff$ $G \curvearrowright \hat{\mathcal{A}}$ is free;
- has the **almost extension property** of Nagy and Reznikoff $\iff$ $G \curvearrowleft \hat{\mathcal{A}}$ is essentially free;
- has a **unique pseudo-expectation** in the sense of Pitts $\iff$ $G \curvearrowright \mathcal{A}$ is properly outer in the sense of Kishimoto;
- has a **unique conditional expectation** $\iff$ $G \curvearrowright \mathcal{A}$ is freely acting.
Examples

Example

The $C^*$-inclusion $C(\mathbb{T}) \subseteq C(\mathbb{T}) \rtimes_{\theta} \mathbb{Z}$ (irrational rotation) has the PEP.

Example

The $C^*$-inclusion $\mathcal{O}_2 \subseteq \mathcal{O}_2 \rtimes_{\alpha} \mathbb{Z}_2$ (switch the generators) has the AEP but not the PEP.

Example

The $C^*$-inclusion $C(\mathbb{T}) \subseteq C(\mathbb{T}) \rtimes O(2)$ has a unique pseudo-expectation but not the AEP.

Example

The $C^*$-inclusion $K(\ell^2)^1 \subseteq K(\ell^2)^1 \rtimes_{\alpha} \mathbb{Z}_2$ where $\alpha = \text{Ad}(U)$ and

$$U = \bigoplus_{n=1}^{\infty} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in B(\ell^2),$$

has a unique conditional expectation but infinitely many pseudo-expectations parameterized by the interval $[-1, 1]$. 
References


Thanks

Thanks!

Questions?
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Conclusion

Personal Dick Kadison stories

Aug. 2000  MSRI (Clay Mathematics Institute Introductory Workshop in Operator Algebras)
May 2003  GPOTS  Illinois
Oct. 2004  ECOAS  USNA
Mar. 2011  Penn Analysis Seminar - “Toeplitz CAR Flows” by Izumi