CHAPTER

4

Applications of the Derivative

4.1 Linear Approximation and Applications

Preliminary Questions

1. How large is the difference $g(1) - g(1.2)$ (approximately) if $g'(1) = 4$?

2. Estimate $f(2.1)$, assuming that $f(2) = 1$, $f'(2) = 3$.

3. Can you estimate $f(2)$, assuming that $f(2.1) = 4$, $f'(2) = 1$?

4. The instantaneous velocity of a train at a given instant is 110 ft/sec. How far does the train travel during the next half-second (use the Linear Approximation)?

5. Discuss how the Linear Approximation makes the following statement more precise: *the sensitivity of the output to a small change in input depends on the derivative*.

Exercises

In Exercises 1–6, calculate the actual change Δf and the Linear Approximation $f'(a)\Delta x$.

1. x^4, $a = 2$, $\Delta x = .3$

Let $f(x) = x^4$, $a = 2$, and $\Delta x = .3$. Then $f'(x) = 4x^3$ and $f'(a) = f'(2) = 32$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = 32(.3) = 9.6$.
- The actual change is
  $\Delta f = f(a + \Delta x) - f(a) = f(2.3) - f(2) = 27.9841 - 16 = 11.9841$.

3. $\sqrt{1+x}$, $a = 8$, $\Delta x = 1$

Let $f(x) = (1+x)^{1/2}$, $a = 8$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{2}(1+x)^{-1/2}$ and $f'(a) = f'(8) = \frac{1}{6}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{6}(1) = \frac{1}{6} \approx .166667$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(9) - f(8) = \sqrt{10} - 3 \approx .162278$.

5. $\tan x, \quad a = \frac{\pi}{4}, \Delta x = .013$

Let $f(x) = \tan x$, $a = \frac{\pi}{4}$, and $\Delta x = .013$. Then $f'(x) = \sec^2 x$ and $f'(a) = f'(\frac{\pi}{4}) = 2$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = 2(.013) = .026$.
- The actual change is
  $\Delta f = f(a + \Delta x) - f(a) = f(\frac{\pi}{4} + .013) - f(\frac{\pi}{4}) \approx 1.026344 - 1 = .026344$.

7. Use the Linear Approximation for $f(x) = \sqrt{x}$ at $a = 25$ to estimate $\sqrt{26} - \sqrt{25}$. Use a calculator to compute the error in this estimate.

Let $f(x) = \sqrt{x}$, $a = 25$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = f'(25) = \frac{1}{10}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{10}(1) = .1$.
- The actual change is $\Delta f = f(a + \Delta x) - f(a) = f(26) - f(25) \approx .0990195$.
- The error in this estimate is $|.0990195 - .1| = .000980486$.

9. Use the Linear Approximation to estimate $16.5^{1/4} - 16^{1/4}$. Use a calculator to compute the error in this estimate.

Let $f(x) = x^{1/4}$, $a = 16$, and $\Delta x = .5$. Then $f'(x) = \frac{1}{4}x^{-3/4}$ and $f'(a) = f'(16) = \frac{1}{32}$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{32}(.5) = .015625$.
- The actual change is
  $\Delta f = f(a + \Delta x) - f(a) = f(16.5) - f(16) \approx 2.015445 - 2 = .015445$ which gives an error of $|.015625 - .015445| \approx .00018$.

11. Use the Linear Approximation to $\sin x$ at $a = 0$ to estimate $\sin(.023)$.

Let $f(x) = \sin x$, $a = 0$, and $\Delta x = .023$. Then $f'(x) = \cos x$ and $f'(a) = f'(0) = 1$ and note that $\sin(.023) = \sin(.023) - 0 = f(a + \Delta x) - f(a)$. The Linear Approximation to $\sin(.023)$ is $\Delta f \approx f'(a)\Delta x = 1(.023) = .023$.

13. The cube root of 27 is 3. How much larger is the cube root of 27.2? Estimate using the Linear Approximation.

Let $f(x) = x^{1/3}$, $a = 27$, and $\Delta x = .2$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = f'(27) = \frac{1}{27}$. The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{1}{27}(.2) = .0074074$.

15. Approximate $\sin(\frac{\pi}{4} + .01)$ using that $\sin \frac{\pi}{4} = \cos \frac{\pi}{4} \approx .7071$.

Let $f(x) = \sin x$, $a = \frac{\pi}{4}$, and $\Delta x = .01$. Then $f'(x) = \cos x$ and $f'(a) = f'(\frac{\pi}{4}) = \frac{\sqrt{2}}{2} \approx .7071$.

- The Linear Approximation is $\Delta f \approx f'(a)\Delta x = \frac{\sqrt{2}}{2}(.01) = .007071$.
- The actual change is
  $\Delta f = f(a + \Delta x) - f(a) = f(\frac{\pi}{4} + .01) - f(\frac{\pi}{4}) \approx \sin(\frac{\pi}{4} + .01) - \sin(\frac{\pi}{4}) \approx .007036$.

17. Let $f(x) = \sqrt{x}$ and $a = 16$.

(a) Find the linearization $L(x)$ to $f(x) = \sqrt{x}$ at the point $a = 16$.

(b) Use $L(x)$ to compute the approximate value of $\sqrt{16.2}$. Compare with the value obtained from a calculator.

Let $f(x) = x^{1/2}$, $a = 16$, and $\Delta x = .2$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = f'(16) = \frac{1}{8}$.

(a) The linearization to $f(x)$ is $L(x) = f'(a)(x-a) + f(a) = \frac{1}{8}(x-16) + 4 = \frac{1}{8}x + 2$.

(b) We have $L(16.2) = 4.025$ compared with $f(16.2) \approx 4.024922$.

19. Compute the linearization of $f(x) = \frac{1}{\sqrt{x}}$ at $a = 16$ and use it to approximate $\frac{1}{\sqrt{15}}$.

Let $f(x) = \frac{1}{\sqrt{x}}$ and $a = 16$. Then $f'(x) = -\frac{1}{2}x^{-3/2}$ and $f'(a) = f'(16) = -\frac{1}{128}$. The linearization is $L(x) = f'(a)(x-a) + f(a) = -\frac{1}{128}(x-16) + \frac{1}{4}$. Then $\frac{1}{\sqrt{15}} = L(15) = -\frac{1}{128}(-1) + \frac{1}{4} = \frac{33}{128} = .257813$.

In Exercises 21–29, use the linearization to approximate the quantity and compare the result with the value obtained from a calculator.

21. $\sqrt{17}$

Let $f(x) = x^{1/2}$, $a = 16$, and $\Delta x = 1$. Then $f'(x) = \frac{1}{2}x^{-1/2}$ and $f'(a) = f'(16) = \frac{1}{8}$.

- The linearization to $f(x)$ is $L(x) = f'(a)(x-a) + f(a) = \frac{1}{8}(x-16) + 4 = \frac{1}{8}x + 2$.
- We have $L(17) = 4.125$, compared with $f(17) \approx 4.123106$.

23. $(27.03)^{1/3}$

Let $f(x) = x^{1/3}$, $a = 27$, and $\Delta x = .03$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = f'(27) = \frac{1}{27}$.

- The linearization to $f(x)$ is $L(x) = f'(a)(x-a) + f(a) = \frac{1}{27}(x-27) + 3 = \frac{1}{27}x + 2$.
- We have $L(27.03) \approx 3.0011111$, compared with $f(27.03) \approx 3.0011107$.

25. $(27.001)^{1/3}$

Let $f(x) = x^{1/3}$, $a = 27$, and $\Delta x = .001$. Then $f'(x) = \frac{1}{3}x^{-2/3}$ and $f'(a) = f'(27) = \frac{1}{27}$.

- The linearization to $f(x)$ is $L(x) = f'(a)(x-a) + f(a) = \frac{1}{27}(x-27) + 3 = \frac{1}{27}x + 2$.
- We have $L(27.001) \approx 3.0000370370370$, compared with $f(27.001) \approx 3.0000370365798$.

27. $\frac{1}{5.1}$

Let $f(x) = x^{-1}$, $a = 5$, and $\Delta x = .1$. Then $f'(x) = -x^{-2}$ and $f'(a) = f'(5) = -\frac{1}{25}$.

- The linearization to $f(x)$ is $L(x) = f'(a)(x-a) + f(a) = -\frac{1}{25}(x-5) + \frac{1}{5} = \frac{2}{5} - \frac{1}{25}x$.
- We have $L(5.1) = .196$, compared with $f(5.1) \approx .19608$.

29. $(31)^{3/5}$

Let $f(x) = x^{3/5}$, $a = 32$, and $\Delta x = -1$. Then $f'(x) = \frac{3}{5}x^{-2/5}$ and $f'(a) = f'(32) = \frac{3}{20}$.

- The linearization to $f(x)$ is $L(x) = f'(a)(x-a) + f(a) = \frac{3}{20}(x-32) + 8 = \frac{3}{20}x + \frac{16}{5}$.
- We have $L(31) = 7.85$, compared with $f(31) \approx 7.84905$.

31. If a stone is tossed vertically in the air with an initial velocity v ft/sec, then it reaches a maximum height of $v^2/64$ feet.

(a) Use the Linear Approximation to estimate the effect on the maximum height of a change in initial velocity from 25 to 26 ft/sec.

(b) Use the Linear Approximation to estimate the effect on the maximum height of a change in initial velocity from 30 to 31 ft/sec.

(c) Does a one foot per second increase in initial velocity have a bigger effect at low velocities or at high velocities? Explain.

(d) Does a one foot per second increase in initial velocity have a bigger percentage effect at low velocities or at high velocities?

A stone tossed vertically with initial velocity v ft/s attains a maximum height of $h = v^2/64$ ft.

(a) If $v = 25$ and $\Delta v = 1$, then $\Delta h \approx h'(v)\Delta v = \frac{1}{32}(25)(1) = .78125$ ft.

(b) If $v = 30$ and $\Delta v = 1$, then $\Delta h \approx h'(v)\Delta v = \frac{1}{32}(30)(1) = .9375$ ft.

(c) A one foot per second increase in initial velocity v increases the maximum height by $v/32$ ft. Accordingly, there is a bigger effect at higher velocities.

(d) The percentage increase on maximum height of a one foot per second increase in initial velocity v is $\dfrac{v/32}{v} = \dfrac{1}{32}$. Thus there is the same effect at all velocities.

In Exercises 33–34, use the following fact derived from Newton's laws: an object that is released with initial velocity v and angle θ, travels a total distance of $\frac{1}{32}v^2 \sin 2\theta$ feet.

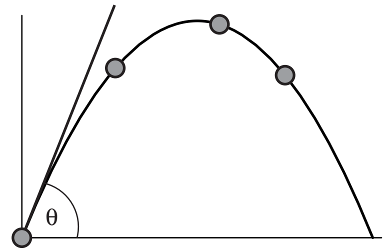


Figure 1 Trajectory of an object released at an angle θ.

33. A player located 18.1 feet from the basket launches a successful jump shot from a height of 10 feet (level with the rim of the basket), at an angle of 34° and initial velocity of 25 ft/s.

(a) Show that the distance of the shot changes by approximately $0.255\Delta\theta$ feet if the angle is changed by an amount $\Delta\theta$ (degrees).

(b) Would the shot likely have been successful if the angle were off by 2°?

Using Newton's laws, given initial velocity $v = 25$ ft/s, the shot travels $x = \frac{1}{32}v^2 \sin 2t = \frac{625}{32} \sin 2t$ ft, where t is in radians.

(a) If $\theta = 34°$ (i.e., $t = \frac{17}{90}\pi$), then $\Delta x \approx x'(t)\Delta t = \frac{625}{16}\cos(\frac{17}{45}\pi)\Delta t = \frac{625}{16}\cos(\frac{17}{45}\pi)\Delta\theta \cdot \frac{\pi}{180} \approx 0.255\Delta\theta$.

(b) If $\Delta\theta = 2°$, this gives $\Delta x \approx 0.51$ ft, in which case the shot would not have been successful, having been off half a foot.

Exercises 35–36 are based on Example **??**.

35. **(a)** Estimate the weight loss per mile of altitude for a 130-lb pilot.

(b) Estimate the altitude at which she would weigh 127 lb.

From the discussion in the text, the weight loss ΔW at altitude h (in miles) for a person weighing W_0 at the surface of the earth is approximately $\Delta W \approx -.0005W_0h$.

(a) The astronaut weighs $W_0 = 130$ lb at the surface of the earth. Accordingly, her weight loss at altitude h is approximately $\Delta W \approx -.065h$.

(b) An estimate of the altitude at which she would weigh 127 lb is given by

$$h \approx \frac{\Delta W}{-.065} = \frac{-3}{-.065} \approx 46.15 \text{ miles.}$$

37. The *stopping distance* for an automobile (after applying the brakes) is approximately $F(s) = 1.1s + .054s^2$ where s is the speed in mph.

(a) Find the Linear Approximation to F at $s = 35$ and $s = 55$.

(b) Use (a) to estimate the change in stopping distance per additional mph for $s = 35$ and $s = 55$.

Let $F(s) = 1.1s + .054s^2$.

(a)
- The Linear Approximation at $s = 35$ mph is $\Delta F \approx F'(35)\Delta s = (1.1 + .108 \times 35)\Delta s = 4.88\Delta s$ ft.
- The Linear Approximation at $s = 55$ mph is $\Delta F \approx F'(55)\Delta s = (1.1 + .108 \times 55)\Delta s = 7.04\Delta s$ ft.

(b)
- The change in stopping distance per additional mph for $s = 35$ mph is approximately 4.88 ft.
- The change in stopping distance per additional mph for $s = 55$ mph is approximately 7.04 ft.

39. The length s of a side of a box in the shape of a cube is measured at 2 feet and the volume $V = s^3$ is then estimated to be 8 ft^3.

(a) Estimate the error in the volume calculation if the measurement of s is inaccurate by $\pm.2$ ft.

(b) What is the maximum allowable error in the measurement of s in inches, if the volume calculation is to have an error of at most 1 ft^3?

The volume V of a cube having side length s is $V = s^3$.

(a) For $s = 2$ ft and $\Delta s = \pm.2$ ft, we have $\Delta V \approx V'(s)\Delta s = 3s^2\Delta s = 12(\pm.2)$ ft^3 $= \pm 2.4$ ft^3.

(b) If the volume computation is to have an error of at most 1 ft^3, then $\Delta V = 12\Delta s = 1$ ft^3, which gives $\Delta s = \frac{1}{12}$ ft $= 1$in.

Further Insights and Challenges

41. Show that for any real number k, $(1 + x)^k \approx 1 + kx$ for small x. Estimate $(1.02)^{.7}$ and $(1.02)^{-.3}$.

Let $f(x) = (1 + x)^k$. Then for small x, we have

$$f(x) \approx L(x) = f'(0)(x - 0) + f(0) = k(1 + 0)^{k-1}(x - 0) + 1 = 1 + kx$$

- Let $k = .7$ and $x = .02$. Then $L(.02) = 1 + (.7)(.02) = 1.104$.
- Let $k = -.3$ and $x = .02$. Then $L(.02) = 1 + (-.3)(.02) = .994$.

4.2 Extreme Values

Preliminary Questions

1. The following questions refer to Figure 1.

(a) How many critical points does $f(x)$ have?

(b) What is the maximum value of $f(x)$ on [0, 8]?

(c) What are the local maximum values of $f(x)$?

(d) Find an interval on which both the minimum and maximum values of $f(x)$ occur at critical points.

(e) Find an interval on which the minimum value occurs at an endpoint.

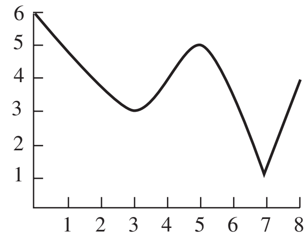


Figure 1

2. Which of the following statements is correct?

(a) If $f(x)$ is not continuous on [0, 1], then it has no minimum or maximum value on [0, 1].

(b) If $f(x)$ is not continuous on [0, 1], then it might not have a minimum or maximum value on [0, 1].

3. Which of the following statements is correct?

(a) A continuous function $f(x)$ does not have a minimum value on the open interval $(0, 1)$.

(b) A continuous function $f(x)$ might not have a minimum value on the open interval $(0, 1)$.

4. State whether true or false and explain why.

(a) The function $f(x) = \frac{1}{x^2}$ has no minimum or maximum value on the interval $(0, 2)$.

(b) The function $f(x) = \frac{1}{x^2}$ has no minimum or maximum value on the interval $(1, 2)$.

(c) The function $f(x) = \frac{1}{x^2}$ has no minimum or maximum value on the interval [1, 2].

5. Which of the following statements is correct?

(a) If $f(x)$ has no critical points in [0, 1], then it has no minimum or maximum value on [0, 1].

(b) If $f(x)$ has no critical points in [0, 1], then either $f(0)$ or $f(1)$ is the minimum value of f on [0, 1].

6. Let $f(x) = ax + b$ be a linear function with positive slope ($a > 0$). State whether true or false and explain why.

(a) $f(x)$ has no critical points.

(b) f has neither a minimum nor a maximum value on a closed interval $[c, d]$.

(c) The maximum value of $f(x)$ on [0, 1] is $a + b$.

Exercises

1. Let $f(x) = 3x^2 - 12x + 3$.

(a) Find the critical point c of $f(x)$ and compute $f(c)$.

(b) Compute the value of $f(x)$ at the endpoints of the interval [0, 4].

(c) Determine the extreme values (the minimum and maximum) of $f(x)$ on [0, 4].

(d) Find the extreme values of $f(x)$ on [0, 1].

Let $f(x) = 3x^2 - 12x + 3$.

(a) Then $f'(c) = 6c - 12 = 0$ implies that $c = 2$ is the sole critical point of f. We have $f(2) = -9$.

(b) Since $f(0) = f(4) = 3$, the maximum value of f on [0, 4] is 3 and the minimum value is -9.

(c) We have $f(1) = -6$. Hence the maximum value of f on [0, 1] is 0 and the minimum value is -6.

In Exercises 3–10, find all critical points of the following functions.

3. $x^2 - 2x + 4$

Let $f(x) = x^2 - 2x + 4$. Then $f'(x) = 2x - 2 = 0$ implies that $x = 1$ is the lone critical point of f.

5. $x^3 - \dfrac{9}{2}x^2 - 54x + 2$

Let $f(x) = x^3 - \frac{9}{2}x^2 - 54x + 2$. Then $f'(x) = 3x^2 - 9x - 54 = 3(x+3)(x-6) = 0$ implies that $x = -3$ and $x = 6$ are the critical points of f.

7. $\dfrac{x}{x^2+1}$

Let $f(x) = \dfrac{x}{x^2+1}$. Then $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} = 0$ implies that $x = \pm 1$ are the critical points of f.

9. $x^{1/3}$

Let $f(x) = \frac{1}{3}$. Then $f'(x) = \frac{1}{3}x^{-2/3} = 0$ implies that $x = 0$ is the critical point of f.

11. Let $f(x) = \sin x + \cos x$.

(a) Find the critical points of $f(x)$.

(b) Determine the maximum value of $f(x)$ on $[0, \frac{\pi}{2}]$.

(a) Let $f(x) = \sin x + \cos x$. Then on the interval $\left[0, \frac{\pi}{2}\right]$, we have $f'(x) = \cos x - \sin x = 0$ at $x = \frac{\pi}{4}$, the only critical point of f in this interval.

(b) Since $f(\frac{\pi}{4}) = \sqrt{2}$ and $f(0) = f(\frac{\pi}{2}) = 0$, the maximum value of f on $\left[0, \frac{\pi}{2}\right]$ is $\sqrt{2}$.

In Exercises 13–40, find the maximum and minimum values of the function on the given interval.

13. $2x^2 - 4x + 2; \quad [0, 3]$

Let $f(x) = 2x^2 - 4x + 2$. Then $f'(x) = 4x - 4 = 0$ implies that $x = 1$ is a critical point of f. On the interval $[0, 3]$, the minimum value of f is $f(1) = 0$, whereas the maximum value of f is $f(3) = 8$. (Note: $f(0) = 2$.)

15. $x^2 - 6x - 1; \quad [-2, 2]$

Let $f(x) = x^2 - 6x - 1$. Then $f'(x) = 2x - 6 = 0$, whence $x = 3$ is a critical point of f. The minimum of f on the interval $[-2, 2]$ is $f(2) = -9$, whereas its maximum is $f(-2) = 15$. (*Note:* critical point $x = 3$ is outside the interval.)

17. $-4x^2 + 3x + 4; \quad [-1, 1]$

Let $f(x) = -4x^2 + 3x + 4$. Then $f'(x) = -8x + 3 = 0$, whence $x = \frac{3}{8}$ is a critical point of f. The minimum of f on the interval $[-1, 1]$ is $f(-1) = -3$, whereas its maximum is $f(\frac{3}{8}) = 4.5625$. (Note: $f(1) = 3$.)

19. $x^3 - 6x + 1$; $[-1, 1]$

Let $f(x) = x^3 - 6x + 1$. Then $f'(x) = 3x^2 - 6 = 0$, whence $x = \pm\sqrt{2}$ are the critical points of f. The minimum of f on the interval $[-1, 1]$ is $f(1) = -4$, whereas its maximum is $f(-1) = 6$. (*Note:* critical points $x = \pm\sqrt{2}$ are not in the interval.)

21. $x^3 + 3x^2 - 9x + 2$; $[-1, 1]$

Let $f(x) = x^3 + 3x^2 - 9x + 2$. Then $f'(x) = 3x^2 + 6x - 9 = 3(x + 3)(x - 1) = 0$, whence $x = 1$ and $x = -3$ are the critical points of f. The minimum of f on the interval $[-1, 1]$ is $f(1) = -3$, whereas its maximum is $f(-1) = 13$. (*Note:* the critical point $x = -3$ is not in the interval.)

23. $x^3 + 3x^2 - 9x + 2$; $[-4, 4]$

Let $f(x) = x^3 + 3x^2 - 9x + 2$. Then $f'(x) = 3x^2 + 6x - 9 = 3(x + 3)(x - 1) = 0$, whence $x = 1$ and $x = -3$ are the critical points of f. The minimum of f on the interval $[-4, 4]$ is $f(1) = -3$, whereas its maximum is $f(4) = 78$. (*Note:* $f(-4) = 22$ and $f(-3) = 29$.)

25. $x^5 - 3x^2$; $[-1, 5]$

Let $f(x) = x^5 - 3x^2$. Then $f'(x) = 5x^4 - 6x = 0$, whence $x = 0$ and $x = \frac{1}{5}(150)^{1/3} \approx 1.06$ are critical points of f. Over the interval $[-1, 5]$, the minimum value of f is $f(-1) = -4$, whereas its maximum value is $f(5) = 3050$. (*Note:* $f(0) = 0$ and $f(\frac{1}{5}(150)^{1/3}) = -\frac{9}{125}(150)^{2/3} \approx -2.03$).

27. $\dfrac{x^2 + 1}{x - 4}$; $[5, 6]$

Let $f(x) = \dfrac{x^2 + 1}{x - 4}$. Then $f'(x) = \dfrac{(x - 4) \cdot 2x - (x^2 + 1) \cdot 1}{(x - 4)^2} = \dfrac{x^2 - 8x - 1}{(x - 4)^2} = 0$ implies $x = 4 \pm \sqrt{17}$. Neither critical point lies in the interval $[5, 6]$. In this interval, the minimum of f is $f(5) = 25$, while its maximum is $f(6) = \frac{37}{2} = 18.5$.

29. $x - \dfrac{4x}{x + 1}$; $[0, 3]$

Let $f(x) = x - \dfrac{4x}{x + 1}$. Then $f'(x) = 1 - \dfrac{4x}{(x + 1)^2} = \dfrac{(x - 1)(x + 3)}{(x + 1)^2} = 0$ whence $x = 1$, $x = -1$ and $x = -3$ are critical points of f. The minimum of f on the interval $[0, 3]$ is $f(1) = -1$, whereas its maximum is $f(0) = f(3) = 0$. (Note: critical points $x = 1$ and $x = -3$ are not in the interval.)

31. $(2 + x)\sqrt{2 + (2 - x)^2}$; $[0, 2]$

Let $f(x) = (2 + x)\sqrt{2 + (2 - x)^2}$. Then

$$f'(x) = \sqrt{2 + (2 - x)^2} - (2 + x)(2 + (2 - x)^2)^{-1/2}(2 - x) = \frac{2(x - 1)^2}{\sqrt{2 + (2 - x)^2}} = 0 \text{ whence}$$

$x = 1$ is the critical point of f. In the interval $[0, 2]$, the minimum occurs at $f(1) = 3\sqrt{3} \approx 5.196152$ and the maximum occurs at $f(0) = f(2) = 4\sqrt{2} \approx 5.656854$.

33. $\sqrt{x + x^2} - 2\sqrt{x}$ $[0, 4]$

Let $f(x) = \sqrt{x + x^2} - 2\sqrt{x}$. Then

$$f'(x) = \tfrac{1}{2}(x + x^2)^{-1/2}(1 + 2x) - x^{-1/2} = \frac{1 + 2x - 2\sqrt{1 + x}}{2\sqrt{x}\sqrt{1 + x}} = 0, \text{ whence } x = 0,$$

$x = -1$ and $x = \pm\frac{\sqrt{3}}{2}$ are the critical points of f. On the interval $[0, 4]$, the minimum of f is $f(\frac{\sqrt{3}}{2}) \approx -.589980$ and the maximum is $f(4) \approx .472136$. (Note: $f(0) = 0$ and critical points $x = -1$ and $x = -\sqrt{3}/2$ are not in the interval.)

35. $\sin x \cos x$; $[0, \frac{\pi}{2}]$

Let $f(x) = \sin x \cos x = \frac{1}{2}\sin 2x$. On the interval $\left[0, \frac{\pi}{2}\right]$, $f'(x) = \cos 2x = 0$ when $x = \frac{\pi}{4}$. The minimum of f on this interval is $f(0) = f(\frac{\pi}{2}) = 0$, whereas the maximum is $f(\frac{\pi}{4}) = \frac{1}{2}$.

37. $x + \sin x$; $[0, 2\pi]$

Let $f(x) = x + \sin x$. On the interval $[0, 2\pi]$, $f'(x) = 1 + \cos x = 0$ at $x = \pi$, a critical point of f. The minimum value of f on this interval is $f(0) = 0$, whereas the maximum value over this interval is $f(2\pi) = 2\pi$. (*Note:* $f(\pi) = \pi$.)

39. $\sin^3\theta - \cos^2\theta$; $[0, 2\pi]$

Let $h(\theta) = \sin^3\theta - \cos^2\theta$. On $[0, 2\pi]$, we have $h'(\theta) = 0$ at $\theta = 0, \frac{\pi}{2}, \pi, \pi + \tan^{-1}(\frac{2}{\sqrt{5}}), \frac{3}{2}\pi, 2\pi - \tan^{-1}(\frac{2}{\sqrt{5}})$, and 2π. From the table of function values below, the minimum of h on $[0, 2\pi]$ is -1 and the maximum is 1.

θ	0	$\frac{\pi}{2}$	π	$\pi + \tan^{-1}(\frac{2}{\sqrt{5}})$	$\frac{3}{2}\pi$	$2\pi - \tan^{-1}(\frac{2}{\sqrt{5}})$	2π
$h(\theta)$	-1	1	-1	$-\frac{23}{27}$	-1	$-\frac{23}{27}$	-1

41. Find the critical points of $f(x) = |x - 2|$.

Let $f(x) = |x - 2|$. For $x < 2$, we have $f'(x) = -1$. For $x > 2$, we have $f'(x) = 1$. Now as $x \to 2-$, we have $\dfrac{f(x) - f(2)}{x - 2} = \dfrac{(2 - x) - 0}{x - 2} \to -1$; whereas as $x \to 2+$, we have $\dfrac{f(x) - f(2)}{x - 2} = \dfrac{(x - 2) - 0}{x - 2} \to 1$. Therefore, $f'(2) = \lim_{x\to 2} \dfrac{f(x) - f(2)}{x - 2}$ does not exist and the lone critical point of f is $x = 2$.

43. Find the critical points of $f(x) = |x^2 + 4x - 12|$ (refer to Figure 3 if necessary) and determine the extreme values of $f(x)$ on $[0, 3]$.

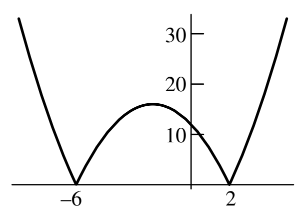
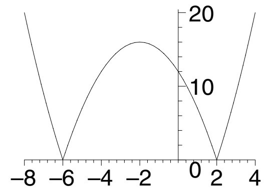


Figure 3

Let $f(x) = |x^2 + 4x - 12| = |(x + 6)(x - 2)|$. From the graph of f below, the critical points of f are $x = -6$ and $x = 2$. On the interval $[0, 3]$ the minimum value of f occurs at $f(2) = 0$ and the maximum value occurs at $f(0) = 12$. (*Note:* $f(3) = 9$ and the critical point $x = -6$ is not in the interval.)



45. Let $a > b > 0$ and set $f(x) = x^a - x^b$ where $a < b$.

(a) Find the maximum value of $f(x)$ on $[0, 1]$ in terms of a and b.

(b) Calculate the maximum value of $f(x) = x^5 - x^{10}$.

(a) Let $f(x) = x^a - x^b$. Then $f'(x) = ax^{a-1} - bx^{b-1}$. Since $a < b$, $f'(x) = x^{a-1}(a - bx^{b-a}) = 0$ implies critical points $x = 0$ and $x = (\frac{a}{b})^{1/(b-a)}$, which is in the interval $[0, 1]$ as $a < b$ implies $\frac{a}{b} < 1$ and consequently $x = (\frac{a}{b})^{1/(b-a)} < 1$. Also, $f(0) = f(1) = 0$ and $a < b$ implies $x^a > x^b$ on the interval $[0, 1]$, which gives $f(x) > 0$ and thus the maximum value of f on $[0, 1]$ is $f((\frac{a}{b})^{1/(b-a)}) = (\frac{a}{b})^{a/(b-a)} - (\frac{a}{b})^{b/(b-a)}$.

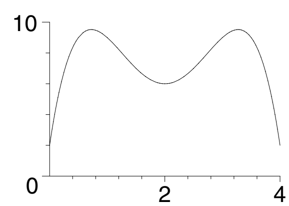
(b) Let $f(x) = x^5 - x^{10}$. Then by part (a), the maximum value of f on $[0, 1]$ is $f((\frac{1}{2})^{1/5}) = (\frac{1}{2}) - (\frac{1}{2})^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$.

47. Sketch the graph of a function on $[0, 4]$ having

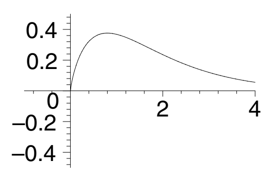
(a) two local maxima and one local minimum

(b) absolute minimum that occurs at an endpoint, and an absolute maximum that occurs at a critical point

(a) Here is the graph of a function on $[0, 4]$ that has two local maxima and one local minimum.

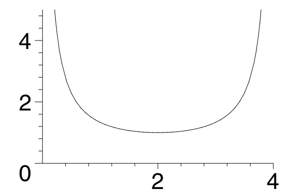


(b) Here is the graph of a function on $[0, 4]$ that has its absolute minimum at its left endpoint and its absolute maximum at an interior critical point.



49. Sketch the graph of a function on $(0, 4)$ with a minimum value but no maximum value.

Here is the graph of a function f on $(0, 4)$ with a minimum value [at $x = 2$] but no maximum value [since $f(x) \to \infty$ as $x \to 0+$].



Further Insights and Challenges

The graph of a quadratic function $f(x) = ax^2 + bx + c$ is a parabola. The function $f(x)$ has a unique absolute minimum value if $a > 0$ and a unique maximum value if $a < 0$.

51. Show that the function $f(x) = x^2 - 2x + 3$ takes on only positive values. *Hint:* consider the minimum value of f.

Observe that $f(x) = x^2 - 2x + 3 = (x - 1)^2 + 2 > 0$ for all x.

53. Figure 4 shows two graphs of cubic polynomials. It shows that a cubic polynomial may have a local minimum and maximum or it may have neither. Find conditions on the coefficients a and b of the cubic polynomial $f(x) = \frac{1}{3}x^3 + \frac{1}{2}ax^2 + bx + c$, which insure that f has neither a local minimum nor a local maximum. *Hint:* apply Exercise 52(a) to $f'(x)$.

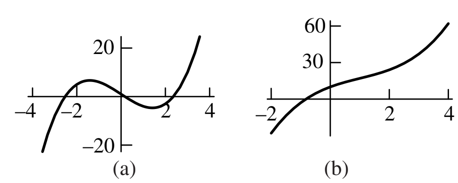


Figure 4 (a) Graph of $x^3 - 6x + 1$. (b) Graph of $x^3 - 3x^2 + 9x + 10$.

Let $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + bx + c$. Using Exercise **??**, we have $g(x) = f'(x) = x^2 + ax + b > 0$ for all x provided $b > \frac{1}{4}a^2$, in which case f has no critical points and hence no local extrema. (Actually $b \geq \frac{1}{4}a^2$ will suffice, since in this case [as we'll see in a later section] f has an inflection point but no local extrema.)

55. Find the minimum and maximum values of $f(x) = x^p(1 - x)^q$ on $[0, 1]$, where p, q are positive numbers.

Let $f(x) = x^p(1 - x)^q$, $0 \leq x \leq 1$, where p and q are positive numbers. Then

$$f'(x) = x^p q(1 - x)^{q-1}(-1) + (1 - x)^q p x^{p-1}$$
$$= x^{p-1}(1 - x)^{q-1}(p(1 - x) - qx) = 0 \quad \text{at} \quad x = \frac{p}{p+q}$$

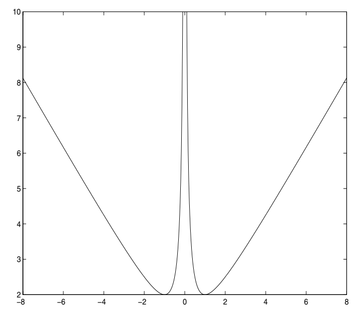
The minimum value of f on $[0, 1]$ is $f(0) = f(1) = 0$, whereas its maximum value is $f(\frac{p}{p+q}) = \frac{p^p}{(p+q)^{p+q}}$.

57. R & W Convince yourself that there does not exist a continuous function with two local minima but no local maximum.

(a) Sketch the graph of a (necessarily discontinuous) function with two local minima but no local maximum.

(b) Let $f(x)$ be a continuous function. Suppose that $f(a)$ and $f(b)$ are local minima where $a < b$. Prove that there exists a value c between a and b such that $f(c)$ is a local maximum. *Hint:* Apply Theorem **??** to the interval $[a, b]$.

(a) The function graphed here is discontinuous at $x = 0$.

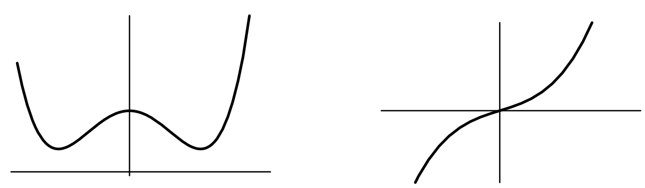


(b) Let $f(x)$ be a continuous function with $f(a)$ and $f(b)$ local minima on the interval $[a, b]$. By Theorem 1, $f(x)$ must take on both a minimum and a maximum on $[a, b]$. Since local minima occur at $f(a)$ and $f(b)$, the maximum must occur at some other point in the interval, call it c, where $f(c)$ is a local maximum.

59. (Adapted from *Calculus Problems for a New Century*.) Use Rolle's Theorem to prove the following statements:

(a) If $f'(x)$ is the function shown in Figure 7(A), then $f(x) = 0$ has at most one solution.

(b) If $f'(x)$ is the function shown in Figure 7(B), then $f(x) = 0$ cannot have two positive solutions.



(A) (B)

Figure 7 Graphs of the derivatives.

(a) Let the graph of $f'(x)$ be as shown in the text; note that $f'(x) > 0$ for all x. Assume that $f(x) = 0$ has two distinct solutions, say a and b. By Rolle's Theorem, there is a number c between a and b such that $f'(c) = 0$, contradicting the fact that f' is everywhere positive. Accordingly, $f(x) = 0$ has at most one solution.

(b) Let the graph of $f'(x)$ be as shown in the text; note that $f'(x) > 0$ for $x > 0$. Assume that $f(x) = 0$ has two distinct positive solutions, say a and b with $a < b$. By Rolle's Theorem, there is a number c with $0 < a < c < b$ such that $f'(c) = 0$, contradicting the fact that $f'(x)$ is positive for positive values of x. Accordingly, $f(x) = 0$ has at most one positive solution.

4.3 The Mean Value Theorem and Monotonicity

Preliminary Questions

1. Which value of m makes the following statement correct? *If $f(2) = 3$ and $f(4) = 9$, where $f(x)$ is differentiable, then the graph must have a tangent line whose slope is m.*

2. Which of the following conclusions does *not* follow from the Mean Value Theorem (assume f is differentiable)?

(a) If f has a secant line whose slope is 0, then $f'(x) = 0$ for some value of x.

(b) If $f'(x) = 0$ for some value of x, then there is a secant line whose slope is 0.

(c) If $f'(x) > 0$ for all x, then every secant line has positive slope.

3. State whether true or false and explain why (assume f is differentiable).

(a) If $f(1) = f(3) = 5$, then $f'(c) = 0$ for at least one value $1 < c < 3$.

(b) If $f'(c) = 0$ for some $c \in [a, b]$, then $f(a) = f(b)$.

(c) If $f(5) < f(9)$, then $f'(c) > 0$ for some $c \in [5, 9]$.

4. Can a function that takes on only negative values have a positive derivative? Give an example or explain why no such functions exist.

5. Which of the following two situations corresponds to a *negative* first derivative? Explain.

(a) A car slowing down.

(b) A car driving in reverse.

6. **(a)** Use the graph of $f'(x)$ in Figure 1 to determine whether $f(c)$ is a local minimum or maximum.

(b) Is it correct to conclude from Figure 1 that $f(x)$ is a decreasing function?

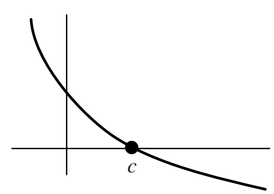


Figure 1 Graph of derivative $f'(x)$.

7. The graph of the derivative $f'(x)$ is shown in Figure 2. Does $f(x)$ have two local minima and one local maximum or does it have two local maxima and one local minimum?

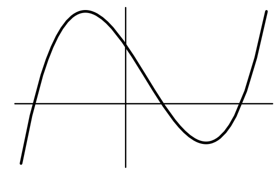


Figure 2 Graph of derivative $f'(x)$.

8. Sam made two statements which Deborah found dubious.

(a) "Although my average velocity for the trip was 70 mph, at no point in time did my speedometer read 70 miles per hour."

(b) "Although a policemen clocked me going 70 mph, my speedometer never read 65 mph."

Which theorem did Deborah apply in each case: the Intermediate Value Theorem or the Mean Value Theorem?

Exercises

1. Determine the intervals on which $f'(x)$ is positive and negative if $f(x)$ is the function graphed in Figure 3.

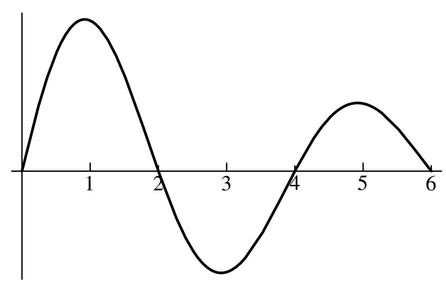


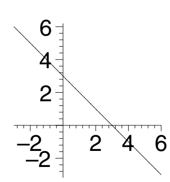
Figure 3

The derivative of f is positive on the intervals (0, 1) and (3, 5); it is negative on the (1, 3) and (5, 6).

In Exercises 2–6, sketch graphs of functions whose derivatives have the following descriptions.

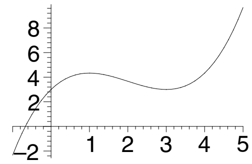
3. f' is negative for all x.

Here is the graph of a function f for which f' is negative for all x.



5. f' is negative on $(1, 3)$ and positive everywhere else.

Here is the graph of a function f for which f' is negative on $(1, 3)$ and positive elsewhere.



In Exercises 7–9, use the First Derivative Test to determine whether the given critical point is a local minimum or local maximum (or neither).

7. $7 + 4x - x^2$; $c = 2$

Let $f(x) = 7 + 4x - x^2$. Then $f'(c) = 4 - 2c = 0$ implies $c = 2$ is a critical point of f. Since f' makes the sign transition $+, -$ as x increases through $c = 2$, we conclude that $f(2) = 11$ is a local maximum of f.

9. $x^3 - 27x + 2$; $c = -3$

Let $f(x) = x^3 - 27x + 2$. Then $f'(c) = 3c^2 - 27 = 0$ at $c = -3$. Since f' makes the sign transition $+, -$ as x increases through $c = -3$, we conclude that $f(-3) = 56$ is a local maximum of f.

11. The graph of the *derivative* of $f(x) = .2x^5 - .125x^4 - .1x^3 - .25x^2 + x$ is shown in Figure 5. Find the critical points of $f(x)$ and determine whether they are local minima, maxima, or neither.

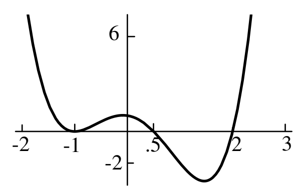


Figure 5 Graph of the *derivative* of $f(x) = .2x^5 - .125x^4 - .1x^3 - .25x^2 + x$.

Let $f(x) = \frac{1}{5}x^5 - \frac{1}{8}x^4 - \frac{1}{10}x^3 - \frac{1}{4}x^2 + x$. Then $f'(x) = x^4 - \frac{1}{2}x^3 - \frac{3}{10}x^2 - \frac{1}{2}x + 1$. Using Maple 8, the roots of f' are all complex. (Their approximate values are $.89 \pm .45i, -.64 \pm .77i$.) This tells us that f' is *nonzero* for all real values of x. Moreover, since f', a polynomial, is everywhere continuous and because $f'(0) = 1 > 0$, we conclude that $f'$is *positive* for all real values of x. Accordingly, there are *no* real critical points of f.

In Exercises 12–18, find a point c satisfying the conclusion of the Mean Value Theorem for the given function and values a, b.

13. $f(x) = \sqrt{x}$, $a = 4$, $b = 9$

Let $f(x) = x^{1/2}$, $a = 4$, $b = 9$. By the MVT, there exists a $c \in (4, 9)$ such that

$$\frac{1}{2}c^{-1/2} = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{3 - 2}{9 - 4} = \frac{1}{5}.$$

Thus $\frac{1}{\sqrt{c}} = \frac{2}{5}$ whence $c = \frac{25}{4} = 6.25 \in (4, 9)$.

15. $f(x) = \cos x - \sin x$, $a = 0$, $b = 2\pi$

Let $f(x) = \cos x - \sin x$, $a = 0$, $b = 2\pi$. By the MVT, there exists a $c \in (0, 2\pi)$ such that

$$-\sin c - \cos c = f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{1 - 1}{2\pi - 0} = 0.$$

Thus $-\sin c = \cos c$. Choose $c = \frac{7}{4}\pi \in (0, 2\pi)$.

17. $f(x) = x^3$, arbitrary a and b

Let $f(x) = x^3$ and let $a < b$ be arbitrary.

- *Case 1*. If $a = 0$, then by the MVT there exists a $c \in (0, b)$ such that $3c^2 = f'(c) = \dfrac{f(b) - f(0)}{b - 0} = \dfrac{b^3 - 0}{b - 0} = b^2$, whence $3c^2 = b^2$ implies $c = \pm\dfrac{b}{\sqrt{3}}$. Choose $c = \dfrac{b}{\sqrt{3}} \in (0, b)$.
- *Case 2*. If $a \neq 0$, consider $g(x) = f(x + a)$ on $[0, B]$, where $A = 0$ and $B = b - a > 0$. By Case 1, $C = \dfrac{B}{\sqrt{3}}$ is such that $g'(C) = \dfrac{g(B) - g(0)}{B - 0}$. By the Chain Rule, we have $g'(x) = f'(x + a)$. Choose

$$c = a + C = a + \frac{B}{\sqrt{3}} = a + \frac{b - a}{\sqrt{3}} \in (a, b).$$

Then

$$\begin{aligned} f'(c) = f'(a + C) = g'(C) &= \frac{g(B) - g(0)}{B - 0} \\ &= \frac{f(B + a) - f(0 + a)}{B - 0} = \frac{f(b) - f(a)}{b - a} \end{aligned}$$

In other words, $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, which is the conclusion of the MVT, as required.

19. Let $f(x) = 1 - |x|$.

(a) Show that $f(x)$ does not satisfy the conclusion of the Mean Value Theorem for the points $a = -2$, $b = 1$.

(b) Why does the Mean Value Theorem not apply in this case?

(c) Show that the Mean Value Theorem does hold if a and b are both positive or both negative.

Let $f(x) = 1 - |x|$.

(a) For $a = -2$ and $b = 1$, we have $\dfrac{f(b) - f(a)}{b - a} = \dfrac{0 - (-1)}{1 - (-2)} = \dfrac{1}{3}$. Yet there is no point $c \in (-2, 1)$ such that $f'(c) = \frac{1}{3}$. Indeed, $f'(x) = 1$ for $x < 0$, $f'(x) = -1$ for $x > 0$, and $f'(0)$ is undefined.

(b) The MVT does not apply in this case, since f is not differentiable on the open interval $(-2, 1)$.

(c) If a and b (where $a < b$) are both positive (or both negative), then f is continuous on $[a, b]$ and differentiable on (a, b). Accordingly, the hypotheses of the MVT are met and the theorem does apply. Indeed, in these cases, any point $c \in (a, b)$ satisfies the conclusion of the MVT (since f' is constant on $[a, b]$ in these instances).

In Exercises 20–35, (1) determine the intervals on which the function is monotonic increasing or decreasing, and (2) use the First Derivative Test to determine if the local extrema are local minima or maxima (or neither).

Here is a table legend for Exercises 20–35.

SYMBOL	MEANING
$-$	The entity is negative on the given interval.
0	The entity is zero at the specified point.
$+$	The entity is positive on the given interval.
U	The entity is undefined at the specified point.
$\nearrow$	f is increasing on the given interval.
$\searrow$	f is decreasing on the given interval.
M	f has a local maximum at the specified point.
m	f has a local minimum at the specified point.
$\neg$	There is no local extremum here.

21. $5x^2 + 6x - 4$

Let $f(x) = 5x^2 + 6x - 4$. Then $f'(x) = 10x + 6 = 0$ yields the critical point $c = -\frac{3}{5}$.

x	$\left(-\infty, -\frac{3}{5}\right)$	$-3/5$	$\left(-\frac{3}{5}, \infty\right)$
f'	$-$	0	$+$
f	$\searrow$	m	$\nearrow$

23. $x(x+1)^3$

Let $f(x) = x(x+1)^3$. Then $f'(x) = x \cdot 3(x+1)^2 + (x+1)^3 \cdot 1 = (4x+1)(x+1)^2 = 0$ yields critical points $c = -1, -\frac{1}{4}$.

x	$(-\infty, -1)$	-1	$(-1, -1/4)$	$-1/4$	$(-1/4, \infty)$
f'	$-$	0	$-$	0	$+$
f	$\searrow$	$\neg$	$\searrow$	m	$\nearrow$

25. $x^2 + (10-x)^2$

Let $f(x) = x^2 + (10-x)^2$. Then $f'(x) = 2x + 2(10-x)(-1) = 4x - 20 = 0$ yields the critical point $c = 5$.

x	$(-\infty, 5)$	5	$(5, \infty)$
f'	$-$	0	$+$
f	$\searrow$	m	$\nearrow$

27. $\dfrac{1}{x^2+1}$

Let $f(x) = \left(x^2+1\right)^{-1}$. Then $f'(x) = -2x\left(x^2+1\right)^{-2} = 0$ yields critical point $c = 0$.

x	$(-\infty, 0)$	0	$(0, \infty)$
f'	$+$	0	$-$
f	$\nearrow$	M	$\searrow$

29. $x^4 + x^3$

Let $f(x) = x^4 + x^3$. Then $f'(x) = 4x^3 + 3x^2 = x^2(4x+3)$ yields critical points $c = 0, -\frac{3}{4}$.

x	$\left(-\infty, -\frac{3}{4}\right)$	$-\frac{3}{4}$	$\left(-\frac{3}{4}, 0\right)$	0	$(0, \infty)$
f'	$-$	0	$+$	0	$+$
f	$\searrow$	m	$\nearrow$	M	$\nearrow$

31. $x^2 - x^4$

Let $f(x) = x^2 - x^4$. Then $f'(x) = 2x - 4x^3 = 2x\left(1 - 2x^2\right) = 0$ yields critical points $c = 0, \pm\frac{1}{\sqrt{2}}$.

x	$\left(-\infty, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{\sqrt{2}}$	$\left(-\frac{1}{\sqrt{2}}, 0\right)$	0	$\left(0, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{\sqrt{2}}$	$\left(\frac{1}{\sqrt{2}}, \infty\right)$
f'	+	0	−	0	+	0	−
f	↗	M	↘	m	↗	M	↘

33. $\cos\theta + \sin\theta$, $[0, 2\pi]$

Let $f(\theta) = \cos\theta + \sin\theta$. Then $f'(\theta) = \cos\theta - \sin\theta$, which yields $c = \frac{\pi}{4}, \frac{5\pi}{4}$ on the interval $[0, 2\pi]$.

x	$\left[0, \frac{\pi}{4}\right)$	$\frac{\pi}{4}$	$\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$	$\frac{5\pi}{4}$	$\left(\frac{5\pi}{4}, 2\pi\right]$
f'	+	0	−	0	+
f	↗	M	↘	m	↗

35. $\theta - 2\cos\theta$, $[0, 2\pi]$

Let $f(\theta) = \theta - 2\cos\theta$. Then $f'(\theta) = 1 + 2\sin\theta$, which yields $c = \frac{7\pi}{6}, \frac{11\pi}{6}$ on the interval $[0, 2\pi]$.

x	$\left[0, \frac{7\pi}{6}\right)$	$\frac{7\pi}{6}$	$\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$	$\frac{11\pi}{6}$	$\left(\frac{11\pi}{6}, 2\pi\right]$
f'	+	0	−	0	+
f	↗	M	↘	m	↗

37. Let $f(x) = x^3 - 2x^2 + 2x$.

(a) Find the minimum value of $f'(x)$.

(b) Use (a) to show that f is an increasing function.

Let $f(x) = x^3 - 2x^2 + 2x$.

(a) For all x, we have $f'(x) = 3x^2 - 4x + 2 = 3\left(x - \frac{2}{3}\right)^2 + \frac{2}{3} \geq \frac{2}{3} > 0$. The minimum value of $f'(x)$ is $f'\left(\frac{2}{3}\right) = \frac{2}{3}$.

(b) Since $f'(x) > 0$ for all x, the function f is everywhere increasing.

Further Insights and Challenges

39. Which values of c satisfy the conclusion of the Mean Value Theorem on the interval $[a, b]$ if $f(x)$ is a linear function?

Let $f(x) = px + q$, where p and q are constants. Then the slope of *every* secant line and tangent line of f is p. Accordingly, considering the interval $[a, b]$, *every* point $c \in (a, b)$ satisfies $f'(c) = p = \dfrac{f(b) - f(a)}{b - a}$, the conclusion of the MVT.

41. Find a value of c satisfying the conclusion of the Mean Value Theorem for $f(x) = x^n$ on the interval $[0, b]$.

Let $f(x) = x^n$ and consider the interval $[0, b]$. By the MVT we have $nc^{n-1} = f'(c) = \dfrac{f(b) - f(0)}{b - 0} = \dfrac{b^n}{b} = b^{n-1}$. Thus $nc^{n-1} = b^{n-1}$ and hence $c = \dfrac{b}{\sqrt[n-1]{n}} = \dfrac{b}{n^{1/(n-1)}}$.

43. Use the previous exercise to show that the polynomial $f(x) = x^5 + 2x + 9$ has at most one real root.

Let $f(x) = x^5 + 2x + 9$. Then $f'(x) = 5x^4 + 2 > 0$ for all x. By Exercise **??** the graph of f crosses the x-axis at most once. Accordingly, f has at most one real root. (Indeed, its lone root is approximately equal to -1.437.)

45. Prove the following assertion: *if* $f(0) = g(0)$ *and* $f'(x) \le g'(x)$ *for* x, *then* $f(x) \le g(x)$ *for all* x.

Hint: show that $f(x) - g(x)$ is nondecreasing.

- The assertion is *false*. Let $f(x) = 2x$ and $g(x) = 5x$. Then $f(0) = g(0) = 0$. Moreover, $f'(x) = 2 \le 5 = g'(x)$ for all x. But $f(-1) = -2 > -5 = g(-1)$. Therefore, it is *not* true that $f(x) \le g(x)$ for all x. (Note that f and g are odd functions.)
- Suppose instead that $f(0) = g(0)$ and $f'(x) \le g'(x)$ for $x \ge 0$. Let $h(x) = g(x) - f(x)$. Then $h(0) = f(0) - g(0) = 0$ and $h'(x) = g'(x) - f'(x) \ge 0$ for $x \ge 0$. Thus h is nondecreasing. Accordingly, $h(x) = g(x) - f(x) \ge 0$ for $x \ge 0$. Therefore, $f(x) \le g(x)$ for $x \ge 0$.

47. Use Exercise 45 to establish the following assertions for all x. Each assertion follows from the previous one.

(a) $\cos x \ge 1 - \frac{1}{2}x^2$ (use Exercise 46).

(b) $\sin x \ge x - \frac{1}{6}x^3$

(c) $\cos x \le 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

(d) Can you guess the next inequality in the series?

(a) This assertion is *true* because the functions involved are even.

- Let $f(x) = \cos x$ and $g(x) = 1 - \frac{1}{2}x^2$. Then $f(0) = g(0) = 1$ and $f'(x) = -\sin x \ge -x = g'(x)$ for $x \ge 0$ by the alternative formulation of Exercise **??**. Now apply the alternative formulation of Exercise 45 to conclude that $\cos x \ge 1 - \frac{1}{2}x^2$ for $x \ge 0$.
- For $x < 0$, we have $-x > 0$ and thus $\cos x = \cos(-x) \ge 1 - \frac{1}{2}(-x)^2 = 1 - \frac{1}{2}x^2$; i.e., $\cos x \ge 1 - \frac{1}{2}x^2$ for $x < 0$.
- Hence for all x we have $\cos x \ge 1 - \frac{1}{2}x^2$.

(b) This assertion is *false* because the functions involved are odd. We prove the usual alternative instead. Let $f(x) = \sin x$ and $g(x) = x - \frac{1}{6}x^3$. Then $f(0) = g(0) = 0$ and $f'(x) = \cos x \ge 1 - \frac{1}{2}x^2 = g'(x)$ for $x \ge 0$ by part (a). Now apply the alternative formulation of Exercise 45 to conclude that $\sin x \ge x - \frac{1}{6}x^3$ for $x \ge 0$.

(c) This assertion is *true* because the functions involved are even.

- Let $f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $g(x) = \cos x$. Then $f(0) = g(0) = 1$ and $f'(x) = -x + \frac{1}{6}x^2 \ge -\sin x = g'(x)$ for $x \ge 0$ by part (b). Now apply the alternative formulation of Exercise 45 to conclude that $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \ge \cos x$ or $\cos x \le 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ for $x \ge 0$.
- For $x < 0$, we have $-x > 0$ and thus $\cos x = \cos(-x) \le 1 - \frac{1}{2}(-x)^2 + \frac{1}{24}(-x)^4 = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^2$; i.e., $\cos x \le 1 - \frac{1}{2}x^2 + \frac{1}{24}x^2$ for $x < 0$.

■ Hence for all x we have $\cos x \le 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

(d) The next inequality in the series is $\sin x \le x - \frac{1}{6}x^3 + \frac{1}{120}x^5$, valid for $x \ge 0$. We gradually construct so-called Maclaurin series expansions for $\sin x$ and $\cos x$; i.e., $\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ and $\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$. Can you also see from these expansions why $\sin x$ is an odd function and $\cos x$ is an even one?

49. GU Let $f(x) = x^3 \sin(\frac{1}{x})$.

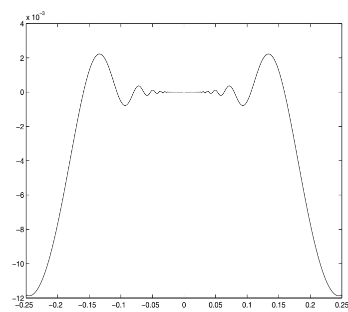
(a) Show that $x = 0$ is a critical point of $f(x)$.

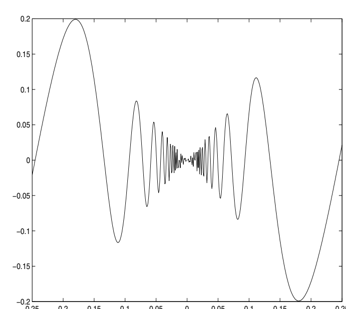
(b) Examine the graphs of $f(x)$ and $f'(x)$. Can the First Derivative Test be applied?

(c) Show that $f(0)$ is neither a local minimum nor maximum.

(a) Let $f(x) = x^3 \sin(\frac{1}{x})$. Then $f'(x) = 3x^2 \sin(\frac{1}{x}) + x^3 \cos(\frac{1}{x})(-x^{-2}) = x(3x \sin(\frac{1}{x}) - \cos(\frac{1}{x}))$, which yields the critical point $x = 0$.

(b) The first figure is $f(x)$ and the second is $f'(x)$. Note how the two functions oscillate near $x = 0$, which implies that the First Derivative Test cannot be applied.





(c) As x approaches 0 from either direction, $f(x)$ alternates between positive and negative, which implies that $f(0)$ alternates between being a local maximum and a local minimum and is therefore considered neither.

4.4 The Shape of a Graph

Preliminary Questions

1. Choose the correct answer: If f is concave up, then f' is:
(a) increasing
(b) decreasing

2. If x_0 is a critical point and f is concave down, then $f(x_0)$ is a:
(a) local minimum
(b) local maximum
(c) cannot be determined

In Questions 3–8, state whether true or false and explain.

3. If $f'(c) = 0$ and $f''(c) < 0$, the $f(c)$ must be a local minimum.

4. A function that is concave down on $(-\infty, \infty)$ can have no minimum value.

5. If $f''(c) = 0$, then f must have a point of inflection at $x = c$.

6. If f has a point of inflection at $x = c$, then $f''(c) = 0$.

7. If f is concave up and f' changes sign at $x = c$, then f' must go from negative to positive at $x = c$.

8. If $f(c)$ is a local maximum, then $f''(c)$ must be negative.

9. Suppose that $f''(c) = 0$ and that $f''(x)$ goes from positive to negative at $x = c$. Which of the following statements are correct?
(a) $f(x)$ has a local maximum at $x = a$.
(b) $f'(x)$ has a local minimum at $x = a$.
(c) $f'(x)$ has a local maximum at $x = a$.
(d) $f(x)$ has a point of inflection at $x = a$.

Exercises

1. Match the graphs in Figure 1 with the description:
(a) $f''(x) < 0$ for all x
(b) $f''(x)$ goes from $+$ to $-$
(c) $f''(x) > 0$ for all x
(d) $f''(x)$ goes from $-$ to $+$

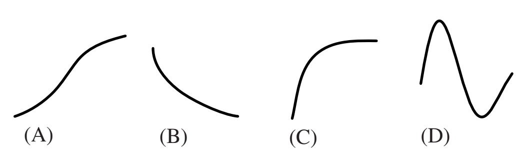


Figure 1

(a) In C, we have $f''(x) < 0$ for all x.
(b) In A, $f''(x)$ goes from $+$ to $-$.

(c) In B, we have $f''(x) > 0$ for all x.
(d) In D, $f''(x)$ goes from $-$ to $+$.

In Exercises 2–7, determine the intervals on which the given function is concave up and concave down and find the points of inflection.

3. $t^3 - 3t^2 + 1$

Let $f(t) = t^3 - 3t^2 + 1$. Then $f'(t) = 3t^2 - 6t$ and $f''(t) = 6t - 6 = 0$ at $t = 1$.

- Thus f is concave up on $(1, \infty)$, since $f''(t) > 0$ there.
- Moreover, f is concave down on $(-\infty, 1)$, since $f''(t) < 0$ there.

5. $(x - 2)(1 - x^3)$

Let $f(x) = (x - 2)\left(1 - x^3\right) = x - x^4 - 2 + 2x^3$. Then $f'(x) = 1 - 4x^3 + 6x^2$ and $f''(x) = 12x - 12x^2 = 12x(1 - x) = 0$ at $x = 0$ and $x = 1$.

- Thus f is concave up on $(0, 1)$ since $f''(x) > 0$ there.
- Moreover, f is concave down on $(-\infty, 0) \cup (1, \infty)$ since $f''(x) < 0$ there.

7. $x^{7/4} - x^2$

Let $f(x) = x^{7/4} - x^2$ and note that f exists for $x \geq 0$ only. Then $f'(x) = \frac{7}{4}x^{3/4} - 2x$ and $f''(x) = \frac{21}{16}x^{-1/4} - 2 = 0$ at $x = \left(\frac{21}{32}\right)^4 \approx .185472$. (*Note:* $f''(x)$ is undefined at $x = 0$.)

- Thus f is concave up on $\left(0, \left(\frac{21}{32}\right)^4\right)$ since $f''(x) > 0$ there.
- Moreover, f is concave down on $\left(\left(\frac{21}{32}\right)^4, \infty\right)$ since $f''(x) < 0$ there.

In Exercises 8–17, find the critical points and apply the Second Derivative Test (if possible) to determine if each critical point is a local minimum or maximum.

9. $x^3 - 3x^2 - 9x$

Let $f(x) = x^3 - 3x^2 - 9x$. Then $f'(x) = 3x^2 - 6x - 9 = 3(x - 3)(x + 1) = 0$ at $x = 3$, -1 and $f''(x) = 6x - 6$. Thus, $f''(3) > 0$, which implies $f(3)$ is a minimum and $f''(-1) < 0$, which implies $f(-1)$ is a maximum.

11. $x^5 - x^2$

Let $f(x) = x^5 - x^2$. Then $f'(x) = 5x^4 - 2x = x(5x^3 - 2) = 0$ at $x = 0, \left(\frac{2}{5}\right)^{\frac{1}{3}}$ and $f''(x) = 20x^3 - 2$. Thus, $f''(0) < 0$, which implies $f(0)$ is a maximum and $f''((\frac{2}{5})^{\frac{1}{3}}) > 0$, which implies $f((\frac{2}{5})^{\frac{1}{3}})$ is a minimum.

13. $\sin^2 x + \cos x$

Let $f(x) = \sin^2 x + \cos x$. Then $f'(x) = 2 \sin x \cos x - \sin x = \sin x(2 \cos x - 1) = 0$ at $x = 0, \pi, 2\pi, \ldots$, and $x = \frac{\pi}{3}, \frac{5\pi}{3}, 7\pi/3, \ldots$, and $f''(x) = 2\cos^2 x - 2\sin^2 x - \cos x$. $f''(x) > 0$ for $x = 0, \pi, 2\pi, \ldots$, which implies that $f(0)$, $f(\pi)$, $f(2\pi)$ and so on are all minima. $f''(x) < 0$ for $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \ldots$, which implies that $f(\frac{\pi}{3})$, $f(\frac{5\pi}{3})$, $f(\frac{7\pi}{3})$ and so on are all maxima.

15. $\dfrac{1}{x^2 - x + 2}$

Let $f(x) = \dfrac{1}{x^2 - x + 2}$. Then $f'(x) = \dfrac{-2x + 1}{(x^2 - x + 2)^2} = \dfrac{-2x + 1}{(x - 2)^2(x - 1)^2} = 0$ at $x = \frac{1}{2}$ and $f''(x) = \dfrac{-2(x - 2)(x - 1) + 2(2x - 1)^2}{(x - 2)^3(x - 1)^3}$. Thus $f''\left(\frac{1}{2}\right) < 0$, which implies that $f\left(\frac{1}{2}\right)$ is a maximum. (*Note:* $f(x)$, $f'(x)$, and $f''(x)$ are undefined at $x = 1, 2$.)

17. $x^{7/4} - x$

Let $f(x) = x^{7/4} - x$. Then $f'(x) = \frac{7}{4}x^{3/4} - 1 = 0$ at $x = (\frac{4}{7})^{4/3}$ and $f''(x) = \frac{21}{16}x^{-1/4}$. Thus, $f''((\frac{4}{7})^{4/3}) > 0$ which implies $f((\frac{4}{7})^{\frac{4}{3}})$ is a minimum.

19. The graph of the derivative $f'(x)$ on [0, 1.2] is shown in Figure 2.

(a) Determine the intervals on which $f(x)$ is increasing and decreasing.

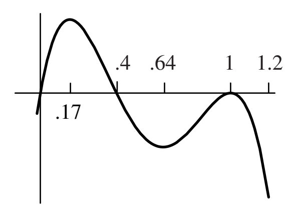


Figure 2

(b) Determine the intervals on which $f(x)$ is concave up and concave down.

(c) Determine the points of inflection of f.

(d) Determine the location of the local minima and maxima of $f(x)$.

Recall that the graph is that of f', *not* f.

(a) f' is increasing on the intervals (0, .17) and (.64, 1) and decreasing on the intervals (.17, .64) and (1, 1.2).

(b)
- Now f is concave up where f' is increasing. This occurs on $(0, .17) \cup (.64, 1)$.
- Moreover, f is concave down where f' is decreasing. This occurs on $(.17, .64) \cup (1, 1.2)$.

(c) The inflection points of f occur where f' changes from increasing to decreasing or vice versa. These occur at $x = .17$, $x = .64$, and $x = 1$.

(d) The local extrema of f occur where f' changes sign. These occur at $x = 0$ and $x = .4$.

In Exercises 20–31, determine the intervals on which the function is concave up and concave down, find the points of inflection, and determine if the critical points are local minima or maxima.

Here is a table legend for Exercises 20–31.

SYMBOL	MEANING
$-$	The entity is negative on the given interval.
0	The entity is zero at the specified point.
$+$	The entity is positive on the given interval.
U	The entity is undefined at the specified point.
$\nearrow$	The function (f, g, etc.) is increasing on the given interval.
$\searrow$	The function (f, g, etc.) is decreasing on the given interval.
$\smile$	The function (f, g, etc.) is concave up on the given interval.
$\frown$	The function (f, g, etc.) is concave down on the given interval.
M	The function (f, g, etc.) has a local maximum at the specified point.
m	The function (f, g, etc.) has a local minimum at the specified point.
I	The function (f, g, etc.) has an inflection point here.
$\neg$	There is no local extremum or inflection point here.

21. $x^2(x-4)$

Let $f(x) = x^2(x-4) = x^3 - 4x^2$.

- Then $f'(x) = 3x^2 - 8x = x(3x-8) = 0$ yields $x = 0, \frac{8}{3}$ as candidates for extrema.
- Moreover, $f''(x) = 6x - 8 = 0$ gives $x = \frac{4}{3}$ as an inflection point candidate.

x	$(-\infty, 0)$	0	$(0, \frac{8}{3})$	$\frac{8}{3}$	$(\frac{8}{3}, \infty)$
f'	$+$	0	$-$	0	$+$
f	$\nearrow$	M	$\searrow$	m	$\nearrow$

x	$(-\infty, \frac{4}{3})$	$\frac{4}{3}$	$(\frac{4}{3}, \infty)$
f''	$-$	0	$+$
f	$\frown$	I	$\smile$

23. $2x^4 - 3x^2 + 2$

Let $f(x) = 2x^4 - 3x^2 + 2$.

- Then $f'(x) = 8x^3 - 6x = 2x\left(4x^2 - 3\right) = 0$ yields $x = 0, \pm\frac{\sqrt{3}}{2}$ as candidates for extrema.
- Moreover, $f''(x) = 24x^2 - 6 = 6(4x^2 - 1) = 0$ gives $x = \pm\frac{1}{2}$ as inflection point candidates.

x	$(-\infty, -\frac{\sqrt{3}}{2})$	$-\frac{\sqrt{3}}{2}$	$(-\frac{\sqrt{3}}{2}, 0)$	0	$(0, \frac{\sqrt{3}}{2})$	$\frac{\sqrt{3}}{2}$	$(\frac{\sqrt{3}}{2}, \infty)$
f'	$-$	0	$+$	0	$-$	0	$+$
f	$\searrow$	m	$\nearrow$	M	$\searrow$	m	$\nearrow$

x	$(-\infty, -\frac{1}{2})$	$-\frac{1}{2}$	$(-\frac{1}{2}, \frac{1}{2})$	$\frac{1}{2}$	$(\frac{1}{2}, \infty)$
f''	$+$	0	$-$	0	$+$
f	$\smile$	I	$\frown$	I	$\smile$

25. $\dfrac{x}{x^2+2}$

Let $f(x) = \dfrac{x}{x^2+2}$.

- Then $f'(x) = \dfrac{2-x^2}{(x^2+2)^2} = 0$ yields $x = \pm\sqrt{2}$ as candidates for extrema.
- Moreover, $f''(x) = \dfrac{-2x\left(x^2+2\right) - (2-x^2)(2)\left(x^2+2\right)(2x)}{(x^2+2)^4} = \dfrac{2x\left(x^2-6\right)}{(x^2+2)^3} = 0$ gives $x = 0, \pm\sqrt{6}$ as inflection point candidates.

x	$\left(-\infty,-\sqrt{2}\right)$	$-\sqrt{2}$	$\left(-\sqrt{2},\sqrt{2}\right)$	$\sqrt{2}$	$\left(\sqrt{2},\infty\right)$
f'	$-$	0	$+$	0	$-$
f	↘	m	↗	M	↘

x	$\left(-\infty,-\sqrt{6}\right)$	$-\sqrt{6}$	$\left(-\sqrt{6},0\right)$	0	$\left(0,\sqrt{6}\right)$	$\sqrt{6}$	$\left(\sqrt{6},\infty\right)$
f''	$-$	0	$+$	0	$-$	0	$+$
f	⌢	I	⌣	I	⌢	I	⌣

27. $\dfrac{1}{x^4+1}$

Let $f(x)=\dfrac{1}{x^4+1}$.

- Then $f'(x)=-\dfrac{4x^3}{(x^4+1)^2}=0$ yields $x=0$ as a candidate for an extremum.
- Moreover,

$$f''(x)=\frac{(x^4+1)^2\left(-12x^2\right)-\left(-4x^3\right)\cdot 2\left(x^4+1\right)\left(4x^3\right)}{(t^2+1)^4}$$
$$=\frac{4x^2\left(5x^4-3\right)}{(x^4+1)^3}=0$$

gives $x=0$ and $x=\left(\frac{3}{5}\right)^{1/4}\approx .880111$ as inflection point candidates.

x	$(-\infty,0)$	0	$(0,\infty)$
g'	$+$	0	$-$
g	↗	M	↘

x	$(-\infty,0)$	0	$(0,.880111)$	$.880111$	$(.880111,\infty)$
g''	$+$	0	$-$	0	$+$
g	⌣	I	⌢	I	⌣

29. $\sin^2 t$ for $0\le t\le \pi$

Let $f(t)=\sin^2 t$ on $[0,\pi]$.

- Then $f'(t)=2\sin t\cos t=\sin 2t=0$ yields $t=\frac{\pi}{2}$ as a candidate for an extremum.
- Moreover, $f''(t)=2\cos 2t=0$ gives $t=\frac{\pi}{4},\frac{3\pi}{4}$ as inflection point candidates.

t	$\left(0,\frac{\pi}{2}\right)$	$\frac{\pi}{2}$	$\left(\frac{\pi}{2},\pi\right)$
f'	$+$	0	$-$
f	↗	M	↘

x	$\left(0,\frac{\pi}{4}\right)$	$\frac{\pi}{4}$	$\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$	$\frac{3\pi}{4}$	$\left(\frac{3\pi}{4},\pi\right)$
f''	$+$	0	$-$	0	$+$
f	⌣	I	⌢	I	⌣

31. $f(x)=\tan x$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$

Let $f(x)=\tan x$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

- Then $f'(x)=\sec^2 x\ge 1>0$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

- Moreover, $f''(x) = 2\sec x \cdot \sec x \tan x = 2\sec^2 x \tan x = 0$ gives $x = 0$ as an inflection point candidate.

x	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
f'	$+$
f	$\nearrow$

x	$\left(-\frac{\pi}{2}, 0\right)$	0	$\left(0, \frac{\pi}{2}\right)$
f''	$-$	0	$+$
f	$\frown$	I	$\smile$

33. R & W An infectious flu spreads slowly at the beginning of an epidemic. The infection process accelerates until a majority of the susceptible individuals are infected, at which point the process slows down.

(a) If $R(t)$ is the number of individuals infected at time t, describe the concavity of the graph of R near the beginning and the end of the epidemic.

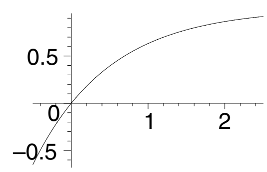
(b) Write a one-sentence news bulletin describing the status of the epidemic on the day that $R(t)$ has a point of inflection.

(a) Near the beginning of the epidemic, the graph of R is concave up. Near the epidemic's end, R is concave down.

(b) "Epidemic subsiding: number of new cases declining."

35. Sketch the graph of a function $f(x)$ such that $f'(x) > 0$ and $f''(x) < 0$ for all x.

Here is the graph of a function $f(x)$ satisfying $f'(x) > 0$ for all x and $f''(x) < 0$ for all x.



37. Sketch the graph of a function $f(x)$ satisfying the conditions:

(i) $f'(x) < 0$ for $x < 1$ and $x > 4$

(ii) $f'(x) > 0$ for $1 < x < 4$

(iii) $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$

There is no function $f(x)$ that satisfies the following conditions.

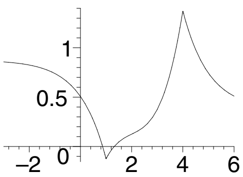
- $f'(x) < 0$ for $x < 1$ and $x > 4$
- $f'(x) > 0$ for $1 < x < 4$
- $f''(x) < 0$ for $x < 2$ and $f''(x) > 0$ for $x > 2$

Suppose such a function exists, and let $g(x) = f'(x)$. Then g satisfies the following conditions.

- $g(x) < 0$ for $x < 1$ and $x > 4$
- $g(x) > 0$ for $1 < x < 4$
- $g'(x) < 0$ for $x < 2$ and $g'(x) > 0$ for $x > 2$

Since $g'(x) > 0$ for $x > 2$, we have that $g(x)$ is continuous for $x > 2$. Since $g(x) > 0$ for $1 < x < 4$ and $g(x) < 0$ for $x > 4$, we must have $g(4) = 0$ by continuity. Hence as x increases through 4, $g(x)$ is decreasing, contradicting the fact that $g'(x) > 0$ for $x > 2$. Therefore, no such function f exists.

However, the following graph of f *almost* fits the bill (except for the fact that f'' does not exist at $x = 1$ or $x = 4$).



Further Insights and Challenges

39. R & W Show that every cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ (with $a \neq 0$) has precisely one point of inflection. Under what conditions (on the coefficients) is the transition at the inflection point from concave up to concave down (instead of from concave down to concave up)?

Let $f(x) = ax^3 + bx^2 + cx + d$, with $a \neq 0$. Then $f''(x) = 6ax + 2b = 0$ implies $f''(-\frac{b}{3a}) = 0$. If $a < 0$, then f'' changes sign from $+$ to $-$, changing from concave up to concave down, as x increases through $-\frac{b}{3a}$. If $a > 0$, then f'' changes sign from $-$ to $+$, changing from concave down to concave up, as x increases through $-\frac{b}{3a}$. In either case, $-\frac{b}{3a}$ is an inflection point for f.

41. Second Derivative Test Give a formal proof of the Second Derivative Test: If c is a critical point such that $f''(c) > 0$, then $f(c)$ is a local minimum. *Hint:* use the fact that the function $F(x)$ in the previous exercise is positive. Assume that $f''(x)$ is continuous, so that $f''(x) > 0$ for all x in a small open interval containing c.

Suppose that $f'(c) = 0$ and $f''(c) = L > 0$. Then

$$f''(c) = \lim_{x\to c} \frac{f'(x) - f'(c)}{x - c} = \lim_{x\to c} \frac{f'(x)}{x - c} = L > 0.$$

Hence there is a $\delta > 0$ such that $0 < |x - c| < \delta$ implies $\dfrac{f'(x)}{x - c} > 0$.

- If $c < x < c + \delta$ (and hence $x - c > 0$), this means $f'(x) > 0$.
- If $c - \delta < x < c$ (and hence $x - c < 0$), this means $f'(x) < 0$.

Accordingly, f' changes sign from $-$ to 0 to $+$ as x increases through c. By the First Derivative Test, $f(c)$ is a local minimum value of f. (The case for a local maximum is similar.)

43. Let $C(x)$ be the cost of producing x units of a certain good. Assume the graph of C is concave up.

(a) Show that the average cost $A(x) = C(x)/x$ is minimized at that production level x_0 for which average cost equals marginal cost.

(b) Explain why the line through $(0, 0)$ and $(x_0, C(x_0))$ is tangent to the graph of $C(x)$.

Let $C(x)$ be the cost of producing x units of a commodity. Assume the graph of C is concave up.

(a) Let $A(x) = C(x)/x$ be the average cost and let x_0 be the production level at which average cost is minimized. Then $A'(x_0) = \dfrac{x_0 C'(x_0) - C(x_0)}{x_0^2} = 0$ implies $x_0 C'(x_0) - C(x_0) = 0$, whence $C'(x_0) = C(x_0)/x_0 = A(x_0)$. In other words, $A(x_0) = C'(x_0)$ or average cost equals marginal cost at production level x_0.

(b) The line between $(0, 0)$ and $(x_0, C(x_0)$ is

$$\begin{aligned}\frac{C(x_0) - 0}{x_0 - 0}(x - x_0) + C(x_0) &= \frac{C(x_0)}{x_0}(x - x_0) + C(x_0)\\ &= A(x_0)(x - x_0) + C(x_0)\\ &= C'(x_0)(x - x_0) + C(x_0)\end{aligned}$$

which is the tangent line to C at x_0.

45. Critical Points and Inflection Points If c is a critical point but $f(c)$ is not a local extreme value, must $x = c$ be a point of inflection? Although this is true of most "reasonable" examples (and it is true of all the examples in the text), this exercise shows that it is not true in general. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

(a) Show that $f'(0)$ exists and is equal to 0. *Hint:* Show that $f'(0) = \lim_{h \to 0} h \sin \frac{1}{h}$ and use the Squeeze Theorem to show that $f'(0) = 0$.

(b) Show that f does not have a local minimum or maximum value at $x = 0$.

(c) Show that f does not have a point of inflection at $x = 0$.

Therefore $x = 0$ is a critical point but $f(0)$ is neither a local extreme value nor a point of inflection.

Let $f(x) = \begin{cases} x^2 \sin(1/x) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$.

(a) Now $f'(0) = \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin\left(\dfrac{1}{x}\right) = 0$ by the Squeeze Theorem: as $x \to 0$ we have

$$\left| x \sin\left(\frac{1}{x}\right) - 0 \right| = |x| \left| \sin\left(\frac{1}{x}\right) \right| \to 0,$$

since the $|\sin u| \leq 1$.

(b) Since $\sin(\frac{1}{x})$ oscillates through every value between -1 and 1 with increasing frequency as $x \to 0$, in any open interval $(-\delta, \delta)$ there are points a and b such that $f(a) = a^2 \sin(\frac{1}{a}) < 0$ and $f(b) = b^2 \sin(\frac{1}{b}) > 0$. Accordingly, $f(0) = 0$ can neither be a local minimum value nor a local maximum value of f.

(c) In part (a) it was shown that $f'(0) = 0$. For $x \neq 0$, we have

$$f'(x) = x^2 \cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) + 2x \sin\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

As $x \to 0$, the difference quotient $\dfrac{f'(x) - f'(0)}{x - 0} = 2 \sin\left(\dfrac{1}{x}\right) - \dfrac{1}{x} \cos\left(\dfrac{1}{x}\right)$ oscillates increasingly rapidly between values of increasingly greater magnitude.

Accordingly, $f''(0) = \lim_{x \to 0} \dfrac{f'(x) - f(0)}{x - 0}$ does not exist. Hence f does not have a point of inflection at $x = 0$.

4.5 Graph Sketching and Asymptotes

Preliminary Questions

1. Match the statement with the appropriate graph and explain.

(a) The outlook is great: the growth rate keeps increasing.
(b) We're losing money, but not as quickly as before.
(c) We're losing money, and it's getting worse as time goes on.
(d) We're doing well but out growth rate is leveling off.
(e) Business had been cooling off but now it's picking up.
(f) Business had been picking up but now it's cooling off.

Figure 1

Exercises

1. Determine the sign combinations of f' and f'' for each interval A–G in Figure 2.

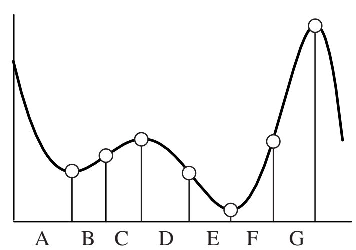


Figure 2

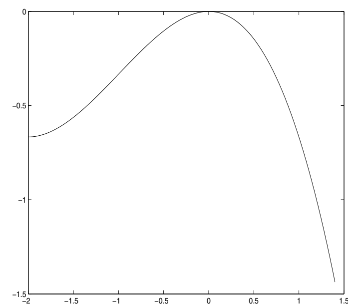
- In A, $f' < 0$ and $f'' > 0$.
- In B, $f' > 0$ and $f'' > 0$.
- In C, $f' > 0$ and $f'' < 0$.

- In D, $f' < 0$ and $f'' < 0$.
- In E, $f' < 0$ and $f'' > 0$.
- In F, $f' > 0$ and $f'' > 0$.
- In G, $f' > 0$ and $f'' < 0$.

In Exercises 3–6, draw the graph of a function for which f' and f'' take on the given sign combinations.

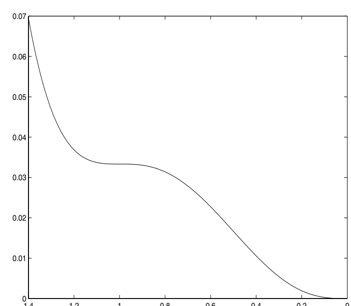
3. $++,\ \ +-,\ \ --$

This function changes from concave up to concave down at $x = -1$ and from increasing to decreasing at $x = 0$.



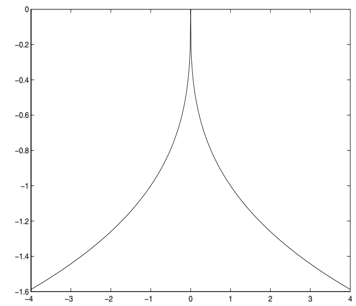
5. $-+,\ \ --,\ \ -+$

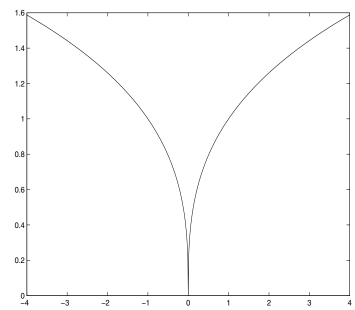
The function is decreasing everywhere and changes from concave up to concave down at $x = -1$ and from concave down to concave up at $x = -\frac{1}{2}$.



7. R & W Are all sign transitions possible for a differentiable function? Explain with a sketch why the sign transitions $++ \rightarrow -+$ and $-- \rightarrow +-$ do not occur if the function is differentiable. (See Ex. 62 for a proof).

In both cases, there is a point where f is not differentiable at the transition from increasing to decreasing or decreasing to increasing.

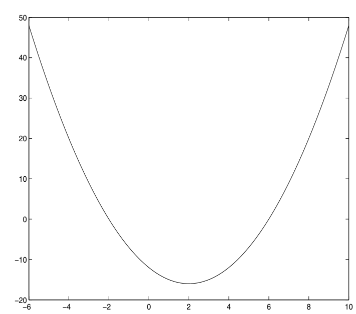




In Exercises 8–10, sketch the graph of the quadratic polynomial.

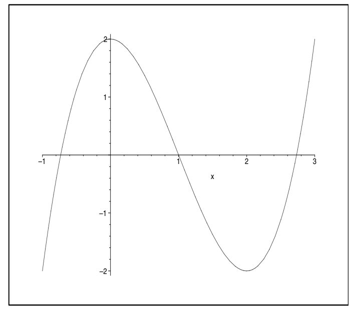
9. $x^2 - 4x - 12$

Let $f(x) = x^2 - 4x - 12$. Then $f'(x) = 2x - 4$ and $f''(x) = 2$. Hence f has a local minimum at $x = 2$ and is concave up everywhere.



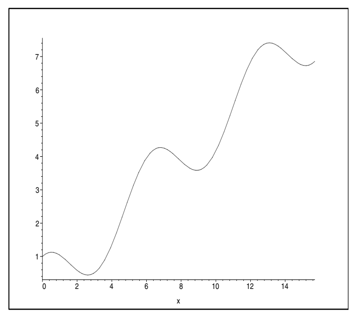
11. Sketch the graph of the cubic $f(x) = x^3 - 3x^2 + 2$. For extra accuracy, plot the zeroes of $f(x)$, which are $x = 1$ and $x = \sqrt{3} \pm 1$ or $x \approx -.73, 2.73$.

Let $f(x) = x^3 - 3x^2 + 2$. Then $f'(x) = 3x^2 - 6x = 3x(x - 2) = 0$ yields $x = 0, 2$and $f''(x) = 6x - 6$. Thus f has an inflection point at $x = 1$, a local maximum at $x = 0$, and a local minimum at $x = 2$.



13. Extend the sketch of the graph of $f(x) = \cos x + \frac{1}{2}x$ over $[0, \pi]$ developed in Example **??** to the interval $[0, 5\pi]$.

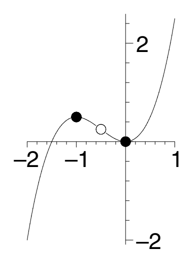
Let $f(x) = \cos x + \frac{1}{2}x$. Then $f'(x) = -\sin x + \frac{1}{2} = 0$ yields critical points at $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}$, and $\frac{29\pi}{6}$ and $f''(x) = -\cos x = 0$ yields $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$, and $\frac{9\pi}{2}$ as points of inflection.



In Exercises 15–24, sketch the graph of the function. Indicate the transition points (local extrema and points of inflection).

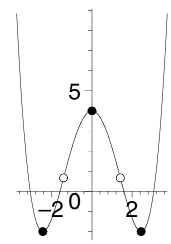
15. $x^3 + \frac{3}{2}x^2$

Let $f(x) = x^3 + \frac{3}{2}x^2$. Then $f'(x) = 3x^2 + 3x = 3x(x + 1)$ and $f''(x) = 6x + 3$. Sign analyses reveal a local maximum at $x = -1$, a local minimum at $x = 0$, and a critical point at $x = -\frac{1}{2}$. Here is a graph of f with these transition points highlighted as in the graphs in the textbook.



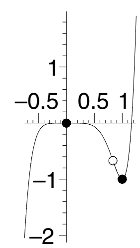
17. $4 - 2x^2 + \frac{1}{6}x^4$

Let $f(x) = \frac{1}{6}x^4 - 2x^2 + 4$. Then $f'(x) = \frac{2}{3}x^3 - 4x = \frac{2}{3}x\left(x^2 - 6\right)$ and $f''(x) = 2x^2 - 4$. Sign analyses reveal local minima at $x = -\sqrt{6}$ and $x = \sqrt{6}$, a local maximum at $x = 0$, and inflection points at $x = \pm\sqrt{2}$. Here is a graph of f with transition points highlighted.



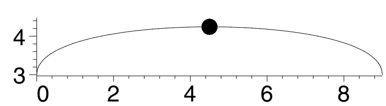
19. $6x^7 - 7x^6$

Let $f(x) = 6x^7 - 7x^6$. Then $f'(x) = 42x^6 - 42x^5 = 42x^5(x - 1)$ and $f''(x) = 252x^5 - 210x^4 = 42x^4(6x - 5)$. Sign analyses reveal a local maximum at $x = 0$, a local minimum at $x = 1$, and an inflection point at $x = \frac{5}{6}$. Here is a graph of f with transition points highlighted.



21. $\sqrt{x} + \sqrt{9 - x}$

Let $f(x) = \sqrt{x} + \sqrt{9 - x} = x^{1/2} + (9 - x)^{1/2}$. Then $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}(9 - x)^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2} - \frac{1}{4}(9 - x)^{-3/2} < 0$ on $(0, 9)$. Sign analyses reveal a local maximum at $x = \frac{9}{2}$. Here is a graph of f with the transition point highlighted.



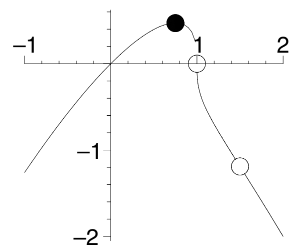
23. $f(x) = x(1 - x)^{1/3}$

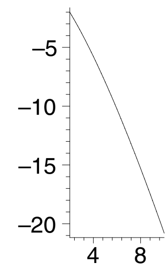
Let $f(x) = x(1 - x)^{1/3}$. Then

$f'(x) = x \cdot \frac{1}{3}(1 - x)^{-2/3}(-1) + (1 - x)^{1/3} \cdot 1 = \dfrac{3 - 4x}{3(x - 1)^{2/3}}$ and similarly

$$f''(x) = \frac{6 - 4x}{9(x - 1)^{5/3}}.$$

Sign analyses reveal a local maximum at $x = \frac{3}{4}$ and inflection points at $x = 1$ and $x = \frac{3}{2}$. Here are two graphs of f with the transition points highlighted.





25. Let $f(x) = 6 + 2(x-3)(x-1)^{2/3}$.

(a) The first two derivatives of $f(x)$ are

$$f'(x) = \frac{\frac{10}{3}(x - \frac{9}{5})}{(x-1)^{1/3}}, \qquad f''(x) = \frac{\frac{20}{9}(x - \frac{3}{5})}{(x-1)^{4/3}}$$

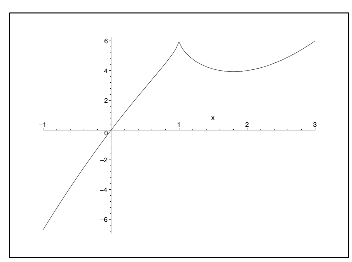
Use this to determine the transition points and the sign combinations of f' and f'' on the intervals between the transition points.

(b) Sketch the graph of $f(x)$.

(a) Let $f(x) = 6 + 2(x-3)(x-1)^{2/3}$. Then $f'(x) = d\frac{\frac{10}{3}(x-\frac{9}{5})}{(x-1)^{1/3}}$ yields critical points at $x = \frac{9}{5}, 1$ and $f''(x) = \dfrac{\frac{20}{9}(x - \frac{3}{5})}{(x-1)^{4/3}}$ yields potential inflection points at $x = \frac{3}{5}, 1$.

Interval	Signs of f' and f''
$\left(-\infty, \frac{3}{5}\right)$	$+-$
$\left(\frac{3}{5}, 1\right)$	$++$
$\left(1, \frac{9}{5}\right)$	$-+$
$\left(\frac{9}{5}, \infty\right)$	$++$

(b) Here is the graph of f.



In Exercises 27–36, calculate the following limits by dividing the numerator and denominator by the highest power of x appearing in the denominator.

27. $\displaystyle\lim_{x\to\infty} \frac{x}{x+9}$

By the Theorem regarding horizontal asymptotes of a rational function (THARF), we have

$$\lim_{x\to\infty} \frac{x}{x+9} = \frac{1}{1} \lim_{x\to\infty} 1 = 1.$$

29. $\displaystyle\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29}$

Via THARF, $\displaystyle\lim_{x\to\infty} \frac{3x^2+20x}{2x^4+3x^3-29} = \frac{3}{2}\lim_{x\to\infty} x^{-2} = 0.$

31. $\displaystyle\lim_{x\to\infty} \frac{7x-9}{4x+3}$

Via THARF, $\displaystyle\lim_{x\to\infty} \frac{7x-9}{4x+3} = \frac{7}{4}\lim_{x\to\infty} 1 = \frac{7}{4}.$

33. $\displaystyle\lim_{x\to-\infty} \frac{7x^2-9}{4x+3}$

Via THARF, $\displaystyle\lim_{x\to-\infty} \frac{7x^2-9}{4x+3} = \frac{7}{4}\lim_{x\to-\infty} x = -\infty.$

35. $\displaystyle\lim_{x\to-\infty} \frac{x^2-1}{x+4}$

Via THARF, $\displaystyle\lim_{x\to-\infty} \frac{x^2-1}{x+4} = \frac{1}{1}\lim_{x\to-\infty} x = -\infty.$

37. Determine which curve in Figure 5 is the graph of $f(x) = (2x^4-1)/(1+x^4)$ on the basis of horizontal asymptotes. Explain your reason.

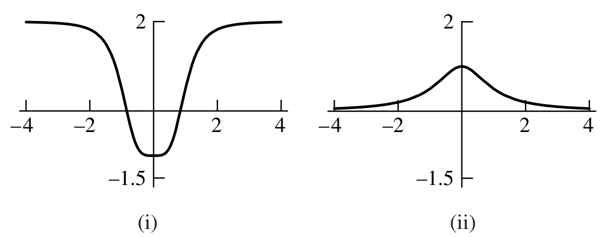


Figure 5

Since (via THARF)

$$\lim_{x\to\pm\infty} \frac{2x^4-1}{1+x^4} = \frac{2}{1}\lim_{x\to\pm\infty} 1 = 2 \quad \text{and} \quad f(0) = -1,$$

the left curve is the picture of the graph of $f(x) = \dfrac{2x^4-1}{1+x^4}$.

39. Match the functions with their graphs in Figure 7.

(a) $\dfrac{1}{x^2-1}$

(b) $\dfrac{x^2}{x^2+1}$

(c) $\dfrac{1}{x^2+1}$

(d) $\dfrac{x}{x^2-1}$

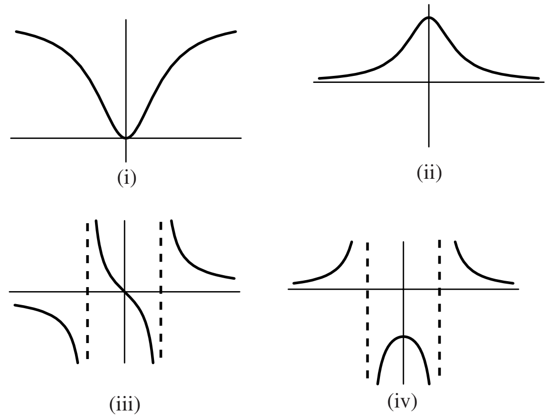


Figure 7

(a) The graph of $1/x^2 - 1$ is (iv).
(b) The graph of $x^2/x^2 + 1$ is (i).
(c) The graph of $1/x^2 + 1$ is (ii).
(d) The graph of $x/x^2 - 1$ is (iii).

41. Let $f(x) = \dfrac{x}{x^2+1}$.

(a) Show that the line $y = 0$ is a horizontal asymptote for f.
(b) Find the critical points of f and the intervals on which f is increasing or decreasing.
(c) Show that

$$f''(x) = \frac{2x(x+\sqrt{3})(x-\sqrt{3})}{(x^2+1)^3}$$

Conclude that the sign of f'' is equal to the sign of the numerator. Make a sign chart and find the intervals on which f'' is positive or negative.

(d) Use the above information to sketch of the graph of f by indicating the behavior as $|x|$ tends to infinity.

Let $f(x) = \dfrac{x}{x^2+1}$.

(a) Via THARF, $\lim_{x\to\infty} f(x) = \frac{1}{1}\lim_{x\to\infty} x^{-1} = 0$. Hence $y = 0$ is a horizontal asymptote for f.

(b) Now $f'(x) = \dfrac{\left(x^2+1\right)\cdot 1 - x\cdot 2x}{\left(x^2+1\right)^2} = \dfrac{1-x^2}{\left(x^2+1\right)^2}$ is negative for $x < -1$ and $x > 1$, positive for $-1 < x < 1$, and 0 at $x = \pm 1$. Accordingly, f has a local minimum value at $x = -1$ and a local maximum value at $x = 1$.

(c) Moreover,

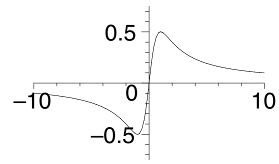
$$f''(x) = \frac{\left(x^2+1\right)^2(-2x) - \left(1-x^2\right)\cdot 2\left(x^2+1\right)\cdot 2x}{(x+1)^4} = \frac{2x^3-6x}{(x+1)^3}$$

$$= \frac{2x\left(x-\sqrt{3}\right)\left(x+\sqrt{3}\right)}{(x+1)^3}.$$

Here is a sign chart for the second derivative, similar to those constructed in various exercises in Section 4.4. (The legend is on page 25.)

x	$\left(-\infty, -\sqrt{3}\right)$	$-\sqrt{3}$	$\left(-\sqrt{3}, 0\right)$	0	$\left(0, \sqrt{3}\right)$	$\sqrt{3}$	$\left(\sqrt{3}, \infty\right)$
f''	$-$	0	$+$	0	$-$	0	$+$
f	$\frown$	I	$\smile$	I	$\frown$	I	$\smile$

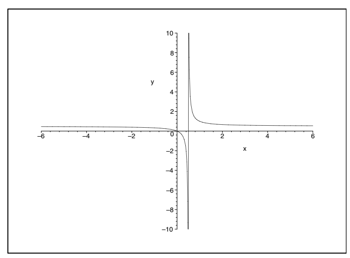
(d) Here is a graph of $f(x) = \dfrac{x}{x^2+1}$.



In Exercises 42–57, sketch the graph of the function. Indicate the asymptotes, local extrema, and points of inflection.

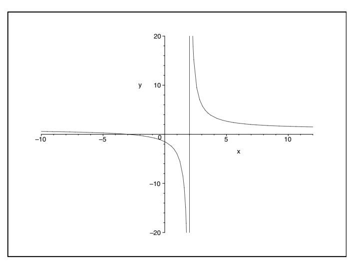
43. $\dfrac{x}{2x-1}$

Let $f(x) = \dfrac{x}{2x-1}$. Then $f'(x) = \dfrac{-1}{(2x-1)^2}$ and $f''(x) = \dfrac{4}{(2x-1)^3}$. Sign analyses reveal no extrema and no points of inflection. Limit analyses give vertical asymptote $x = \frac{1}{2}$ and horizontal asymptote $y = \frac{1}{2}$.



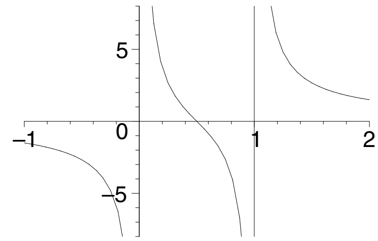
45. $\dfrac{x+3}{x-2}$

Let $f(x) = \dfrac{x+3}{x-2}$. Then $f'(x) = \dfrac{-5}{(x-2)^2}$ and $f''(x) = \dfrac{10}{(x-2)^3}$. Sign analyses reveal no extrema and no points of inflection. Limit analyses give vertical asymptote $x = 2$ and horizontal asymptote $y = 1$.



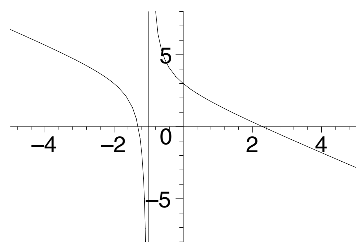
47. $\dfrac{1}{x}+\dfrac{1}{x-1}$

Let $f(x)=\dfrac{1}{x}+\dfrac{1}{x-1}$. Then $f'(x)=-\dfrac{2x^2-2x+1}{x^2(x-1)^2}$ and $f''(x)=\dfrac{2\left(2x^3-3x^2+3x-1\right)}{x^3(x-1)^3}$. Sign analyses reveal an inflection point at $x=\frac{1}{2}$. Limit analyses give vertical asymptotes $x=0$ and $x=1$ and horizontal asymptote $y=0$.



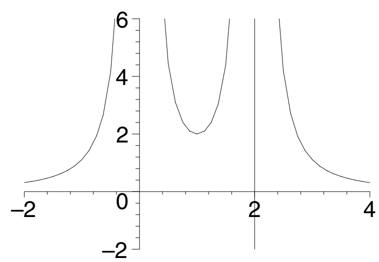
49. $2-x+\dfrac{1}{x+1}$

Let $f(x)=2-x+\dfrac{1}{x+1}$. Then $f'(x)=-\dfrac{x^2+2x+2}{(x+1)^2}=-\dfrac{(x+1)^2+1}{(x+1)^2}$ and $f''(x)=\dfrac{2}{(x+1)^3}$. Sign analyses reveal no transition points. Limit analyses give a vertical asymptote at $x=-1$.



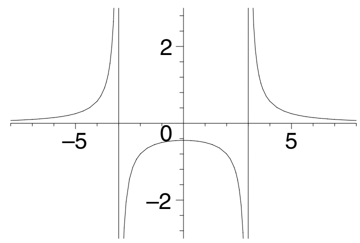
51. $\dfrac{1}{x^2}+\dfrac{1}{(x-2)^2}$

Let $f(x)=\dfrac{1}{x^2}+\dfrac{1}{(x-2)^2}$. Then $f'(x)=-2x^{-3}-2(x-2)^{-3}$ and $f''(x)=6x^{-4}+6(x-2)^{-4}$. Sign analyses reveal a local minimum at $x=1$. Limit analyses give vertical asymptotes $x=0$ and $x=2$ and horizontal asymptote $y=0$.



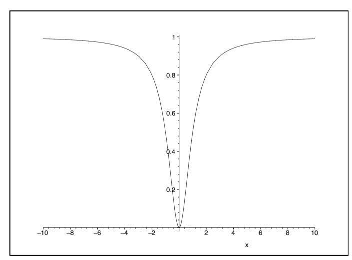
53. $\dfrac{4}{x^2-9}$

Let $f(x)=\dfrac{4}{x^2-9}$. Then $f'(x)=-\dfrac{8x}{(x^2-9)^2}$ and $f''(x)=\dfrac{24\left(x^2+3\right)}{(x^2-9)^3}$. Sign analyses reveal a local maximum at $x=0$. Limit analyses give $x=-3$ and $x=3$ as vertical asymptotes and $y=0$ as a horizontal asymptote.



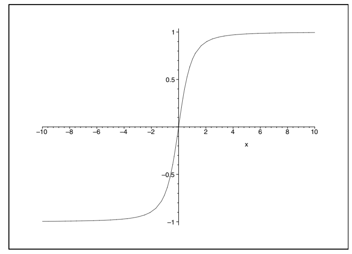
55. $\dfrac{x^2}{(x^2-1)(x^2+1)}$

Let $f(x)=\dfrac{x^2}{(x^2-1)(x^2+1)}$. Then $f'(x)=-\dfrac{2(x+x^5)}{(-1+x^4)^2}$ and $f''(x)=\dfrac{2+24x^4+6x^8}{(-1+x^4)^3}$. Sign analyses reveal a local minimum at $x=0$ and no points of inflection (as the second derivative is always negative).



57. $\dfrac{x}{\sqrt{x^2+1}}$

Let $f(x)=\dfrac{x}{\sqrt{x^2+1}}$. Then $f'(x)=(x^2+1)^{-3/2}$ and $f''(x)=\dfrac{-3x}{(x^2+1)^{\frac{5}{2}}}$. Sign analyses reveal no extrema and a point of inflection at $x=0$. Limit analyses give horizontal asymptotes $y=1$ and $y=-1$.

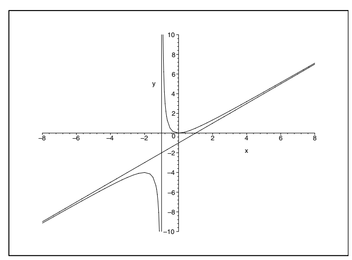


Further Insights and Challenges

Slant Asymptotes The next exercises explore a function whose graph approaches a non-horizontal line as $x \to \infty$. A line $y = ax + b$ is a called a *slant asymptote* if $\lim_{x\to\infty}(f(x) - (ax + b)) = 0$ or $\lim_{x\to-\infty}(f(x) - (ax + b)) = 0$.

59. Sketch the graph of $f(x) = \dfrac{x^2}{x+1}$. Proceed as in the previous exercise (show that $y = x - 1$ is a slant asymptote).

Let $f(x) = \dfrac{x^2}{x+1}$. Then $f'(x) = \dfrac{x(x+2)}{(x+1)^2}$ and $f''(x) = \dfrac{2}{(x+1)^3}$. Sign anaylses reveal a local minimum at $x = 0$, a local maximum at $x = -2$ and that f is concave down on $(-\infty, -1)$ and concave up on $(-1, \infty)$. Limit analyses give a vertical asymptote at $x = -1$. By polynomial division, $f(x) = x - 1 + \dfrac{1}{x+1}$ and $\lim_{x\to\pm\infty} x - 1 + \dfrac{1}{x+1} - (x - 1) = 0$, which implies that the slant asymptote is $x - 1$. Notice that f approaches the slant asymptote as in exercise 59.

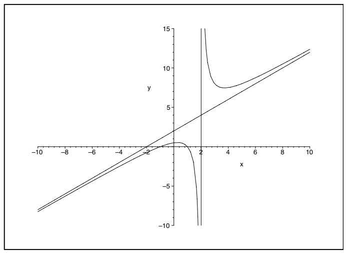


61. Sketch the graph of $f(x) = \dfrac{1-x^2}{2-x}$.

Let $f(x) = \dfrac{1-x^2}{2-x}$. Using polynomial division, $f(x) = x + 2 + \dfrac{3}{x-2}$. Then

$$\lim_{x\to\pm\infty}(f(x) - (x+2)) = \lim_{x\to\pm\infty}(x+2) + \frac{3}{x-2} - (x+2)$$
$$= \lim_{x\to\pm\infty}\frac{3}{x-2} = \frac{3}{1}\lim_{x\to\pm\infty} x^{-1} = 0$$

which implies that $x+2$ is the slant asymptote of $f(x)$. Since $f(x)-(x+2)=\frac{3}{x-2}>0$ as $x\to\infty$, $f(x)$ approaches the slant asymptote from above for $x>2$ and similarly, $\frac{3}{x-2}<0$ as $x\to-\infty$ so $f(x)$ approaches the slant asymptote from below for $x<2$. Then $f'(x)=\frac{x^2-4x+1}{(2-x)^2}$ and $f''(x)=\frac{-6}{(2-x)^3}$. Sign analyses reveal a local minimum at $x=2+\sqrt{3}$, a local maximum at $x=2-\sqrt{3}$ and that f is concave down on $(-\infty,2)$ and concave up on $(2,\infty)$. Limit analyses give a vertical asymptote at $x=2$.



4.6 Applied Optimization

Exercises

1. A 50-inch piece of wire is to be bent into a rectangular shape in such a way as to maximize the area of the rectangle.

(a) Suppose the sides of the rectangle are called x and y. What is the constraint equation relating x and y?

(b) Find a formula for the area in terms of x alone.

(c) Does this problem require optimization over an open interval or a closed interval?

(d) Solve the optimization problem.

(a) The perimeter of the rectangle is 50 inches, which gives $50=2x+2y$, equivalent to $y=25-x$.

(b) Using part (a), $A=xy=x(25-x)=25x-x^2$.

(c) This problem requires an open interval. If $x=0$ or 50 there would not be a rectangle at all.

(d) $A'(x)=25-2x=0$, which yields $x=\frac{25}{2}$ and consequently, $y=\frac{25}{2}$ and the maximum area is $A=156.25$ square inches.

3. The goal is to find the positive number x such that the sum of x and its reciprocal is as small as possible.

(a) Does this problem require optimization over an open interval or a closed interval?

(b) Solve the optimization problem.

Let $x>0$ and $f(x)=x+x^{-1}$.

(a) Here we required optimization over an open interval, since $x \in (0, \infty)$.

(b) Solve $f'(x) = 1 - x^{-2} = 0$ for $x > 0$ to obtain $x = 1$. Since $f''(x) = 2x^{-3} > 0$ for all $x > 0$, f has an absolute minimum of $f(1) = 2$ at $x = 1$.

5. Find positive numbers x, y such that $xy = 16$ and $x + y$ is as small as possible.

Let $x, y > 0$. Now $xy = 16$ implies $y = \frac{16}{x}$. Let $f(x) = x + y = x + 16x^{-1}$. Solve $f'(x) = 1 - 16x^{-2} = 0$ for $x > 0$ to obtain $x = 4$. Since $f''(x) = 32x^{-2} > 0$ for $x > 0$, f has an absolute minimum of $f(4) = 8$ at $x = 4$.

7. Consider the set of all rectangles with area 100.

(a) What are the dimensions of the rectangle with least perimeter?

(b) Is there a rectangle with greatest perimeter? Explain.

Consider the set of all rectangles with area 10.

(a) Let $x, y > 0$ be the lengths of the sides. Now $xy = 100$, whence $y = 100/x$. Let $p(x) = 2x + 2y = 2x + 200x^{-1}$ be the perimeter. Solve $p'(x) = 2 - 200x^{-2} = 0$ for $x > 0$ to obtain $x = 10$. Since $p''(x) = 40x^{-3} > 0$ for $x > 0$, the least perimeter is $p(10) = 40$ when $x = 10$ and $y = 10$.

(b) There is no rectangle in this set with greatest perimeter. For as $x \to \infty$, we have $p(x) = 2x + 20x^{-1} \to \infty$.

9. Suppose that 600 ft of wire are used to build a corral in the shape of a rectangle with a semicircle whose diameter is a side of the rectangle as in Figure 1. Find the dimensions of the corral with maximum area.

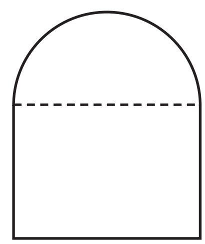


Figure 1

- Let x be the width of the corral and therefore the diameter of the semicircle, and let y be the height of the rectangular section.
- Then the perimeter of the corral can be expressed by the equation $2y + x + \frac{\pi}{2}x = 2y + (1 + \frac{\pi}{2})x = 600$ ft or equivalently, $y = \frac{1}{2}\left(600 - (1 + \frac{\pi}{2})x\right)$.
- The area of the corral is the sum of the area of the rectangle and semicircle, $A = xy + \frac{\pi}{8}x^2$. Making the substitution for y from the constraint equation, $A(x) = \frac{1}{2}x\left(600 - (1 + \frac{\pi}{2})x\right) + \frac{\pi}{8}x^2 = 300x - \frac{1}{2}\left(1 + \frac{\pi}{2}\right)x^2 + \frac{\pi}{8}x^2$.
- $A'(x) = 300 - \left(1 + \frac{\pi}{2}\right)x + \frac{\pi}{4}x = 0$ implies $x = \frac{300}{(1+\frac{\pi}{4})} \approx 168.029746$.
- The optimization is over an open interval and sign analyses reveal that $x = \frac{300}{(1+\frac{\pi}{4})}$ is a local maximum. Thus, the corral of maximum area has dimensions x and $y = 84.01487303$.

11. A rectangular garden of total area 100 ft^2 is to be enclosed with a brick wall costing \$30 per foot on one side and a metal fence costing \$10 per foot on the remaining sides. Find the dimensions of the garden that minimize the cost.

Let x be the length of the brick wall and y the length of an adjacent side with $x, y > 0$. With $xy = 100$ or $y = \frac{100}{x}$, the total cost is $C(x) = 30x + 10(x + 2y) = 40x + 2000x^{-1}$. Solve $C'(x) = 40 - 2000x^{-2} = 0$ for $x > 0$ to obtain $x = 5\sqrt{2}$. Since $C''(x) = 4000x^{-3} > 0$ for all $x > 0$, the minimum cost is $C(5\sqrt{2}) = 400\sqrt{2} \approx \$565.69$ when $x = 5\sqrt{2} \approx 7.07$ ft and $y = 10\sqrt{2} \approx 14.14$ ft.

13. Find the point P on the parabola $y = x^2$ closest to the point $(1, 0)$.

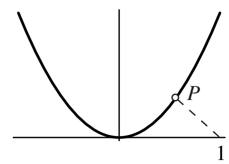


Figure 3

With $y = x^2$, let's equivalently minimize the square of the distance, $f(x) = (x-1)^2 + y^2 = x^4 + x^2 - 2x + 1$. Solve $f'(x) = 4x^3 + 2x - 2 = 0$ to obtain

$$x = x_0 = \frac{(54 + 6\sqrt{87})^{2/3} - 6}{6(54 + 6\sqrt{87})^{1/3}} \approx 0.59$$

(plus two complex solutions, which we discard). Since $f''(x) = 12x^2 + 2 > 0$ for all x, the minimum distance is $\sqrt{f(x_0)} \approx 0.54$ (the exact answer is rather lengthy) when $x = x_0$, $y = x_0^2 \approx 0.35$, and $P \approx (0.59, 0.35)$.

15. Find the dimensions of the rectangle of maximum area that can be inscribed in a circle of radius r.

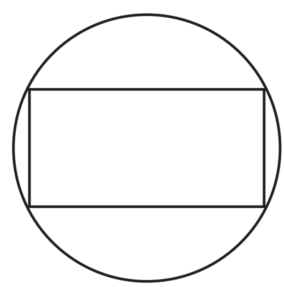


Figure 4

Place the center of the circle at the origin with the sides of the rectangle (of lengths $2x > 0$ and $2y > 0$) parallel to the coordinate axes. By the Pythagorean Theorem, $x^2 + y^2 = r^2$, whence $y = \sqrt{r^2 - x^2}$. Thus the area of the rectangle is $A(x) = 2x \cdot 2y = 4x\sqrt{r^2 - x^2}$. Solve $A'(x) = 4\sqrt{r^2 - x^2} - \dfrac{4x^2}{\sqrt{r^2 - x^2}} = 0$ for $x > 0$ to obtain $x = \frac{r}{\sqrt{2}}$. Since $A''(x) = -\dfrac{12x}{\sqrt{r^2 - x^2}} - \dfrac{4x^3}{(r^2 - x^2)^{3/2}} < 0$ for $x > 0$, the maximum area is $A(\frac{r}{\sqrt{2}}) = 2r^2$ when $x = y = \frac{r}{\sqrt{2}}$.

17. Kepler's Wine Barrel Problem The following problem was stated and solved in a work entitled *Nova stereometria doliorum vinariorum* (Solid Geometry of a Wine Barrel), published in 1615 by the astronomer Johannes Kepler (1571–1630). What are the dimensions of the cylinder of largest volume that can be inscribed in the sphere of radius R? *Hint:* Show that the volume of the cylinder is $2\pi(R^2 - x^2)x$ where x is one-half the height of a cylinder.

Place the center of the sphere at the origin in three-dimensional space. Let the cylinder be of radius x and half-height y. The Pythagorean Theorem states, $x^2 + y^2 = R^2$, whence $x^2 = R^2 - y^2$. The volume of the cylinder is $V(y) = \pi x^2(2y) = 2\pi\left(R^2 - y^2\right)y = 2\pi R^2 y - 2\pi y^3$. Allowing for degenerate cylinders, we have $0 \le y \le R$. Solve $V'(y) = 2\pi R^2 - 6\pi y^2 = 0$ for $y \ge 0$ to obtain $y = \frac{R}{\sqrt{3}}$. Since $V(0) = V(R) = 0$, the largest volume is $V(\frac{R}{\sqrt{3}}) = \frac{4}{9}\pi\sqrt{3}R^3$when $y = \frac{R}{\sqrt{3}}$ and $x = \sqrt{\frac{2}{3}}R$.

19. Find the angle θ that maximizes the area of the quadrilateral with base of length 4 and sides of length 2, as in Figure 5.

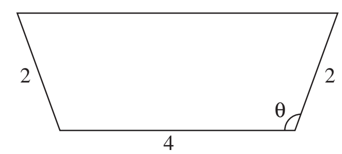


Figure 5

Allowing for degenerate quadrilaterals, we have $0 \le \theta \le \pi$. Via trigonometry and surgery (slice off a right triangle and rearrange the quadrilateral into a rectangle), we have that the area of the quadrilateral is equivalent to the area of a rectangle of base $4 - 2\cos\theta$ and height $2\sin\theta$; i.e, $A(\theta) = (4 - 2\cos\theta)\cdot 2\sin\theta = 8\sin\theta - 4\sin\theta\cos\theta = 8\sin\theta - 2\sin 2\theta$, where $0 \le \theta \le \pi$. Solve $A'(\theta) = 8\cos\theta - 4\cos 2\theta = 4 + 8\cos\theta - 8\cos^2\theta = 0$ for $0 \le \theta \le \pi$ to obtain

$$\theta = \theta_0 = \frac{\pi}{2} + \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \approx 1.94553.$$

Since $A(0) = A(\pi) = 0$, the maximum area is $A(\theta_0) = 3^{1/4}(3 + \sqrt{3})\sqrt{2}$ or approximately 8.80734 when $\theta = \theta_0$.

21. Suppose, in the previous exercise, that the warehouse is to consist of n separate spaces of equal size. Find a formula in terms of n for the maximum possible area of the warehouse.

For n compartments, with x and y as before, cost $= 2{,}400{,}000 = 200((n+1)x + 2ny)$ and $y = \dfrac{12000 - (n+1)x}{2n}$. Then

$$A = nxy = x\frac{12000 - (n+1)x}{2} = 6000x - \frac{n+1}{2}x^2$$

and $A' = 6000 - (n+1)x = 0$ yields $x = \dfrac{6000}{n+1}$ and consequently $y = \dfrac{3000}{n}$. Thus the maximum area is given by

$$A = n\left(\frac{6000}{n+1}\right)\left(\frac{3000}{n}\right) = \frac{18{,}000{,}000}{n+1}.$$

23. According to postal regulations, a carton can be shipped overseas only if the sum of its height and girth does not exceed 74 inches (the girth of a package is the distance around the middle). Find the dimensions of the rectangular carton with the maximum volume, assuming that the base of the carton is a square.

Let the height of the carton be y and the length of a side of its square base be x. The carton's girth is $2x + 2y$. Clearly the volume will be maximized when the sum of the height and girth equals 74; i.e., $2x + 3y = 74$, whence $y = \frac{74}{3} - \frac{2}{3}x$. Allowing for degenerate cartons, the carton's volume is $V(x) = x^2 y = \frac{74}{3}x^2 - \frac{2}{3}x^3$, where $0 \le x \le 37$. Solve $V'(x) = \frac{148}{3}x - 2x^2 = 0$ for x to obtain $x = 0$ or $x = \frac{74}{3}$. Since $V(0) = V(37) = 0$, the maximum volume is $V\left(\frac{74}{3}\right) = 405224/81 \approx 5002.77 \text{ in}^3$ when $x = \frac{74}{3} \approx 24.67$ in and $y = \frac{74}{9} \approx 8.22$ in.

25. A rectangle has dimensions L and W. What is the area of the largest rectangle that can be circumscribed around it? *Hint:* Express the areas of the circumscribed rectangle in terms of the angle θ. Observe that each side of the inner rectangle is the hypotenuse of a right triangle in Figure 7.

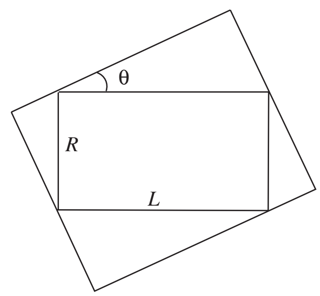


Figure 7

Position the $L \times W$ rectangle in the first quadrant of the xy-plane with its "southwest" corner at the origin. Let θ be the angle the base of the circumscribed rectangle makes with the positive x-axis, where $0 \le \theta \le \frac{\pi}{2}$. Then the area of the circumscribed rectangle is $A = LW + 2 \cdot \frac{1}{2}(W \sin\theta)(W \cos\theta) + 2 \cdot \frac{1}{2}(L \sin\theta)(L \cos\theta) = LW + \frac{1}{2}(L^2 + W^2)\sin 2\theta$, which has a maximum value of $LW + \frac{1}{2}(L^2 + W^2)$ when $\theta = \frac{\pi}{4}$.

27. Given n numbers $x_1, \dots, x_n$, let x be the number for which the sum

$$(x - x_1)^2 + (x - x_2)^2 + \cdots + (x - x_n)^2$$

is minimal. Show that x is equal to the average value

$$\frac{1}{n}(x_1 + x_2 + \cdots + x_n)$$

Let $f(x) = \sum_{k=1}^{n}(x - x_k)^2$. Solve $f'(x) = 2\sum_{k=1}^{n}(x - x_k) = 0$ to obtain $x = \bar{x} = \frac{1}{n}\sum_{k=1}^{n} x_k$. Since $f''(x) = 2\sum_{k=1}^{n} 1 = 2n > 0$ for all x, we conclude that f is minimal when $x = \bar{x}$.

29. Let $\triangle$ be the triangle with vertices $(0,0)$, $(5,0)$, $(0,10)$. Determine the rectangle of largest area whose sides are parallel to the x- and y-axes, which can be placed inside $\triangle$. Use the second derivative test to insure that a minimum has been found.

The line through $(5,0)$ and $(0,10)$ is $y = 10 - 2x$. The area of the rectangle is $A(x) = xy = 10x - 2x^2$. Solve $A'(x) = 10 - 4x = 0$ to obtain $x = \frac{5}{2}$. Since $A''(x) = -4 < 0$ for all x, the minimum area is $A\left(\frac{5}{2}\right) = \frac{25}{2}$ when $x = \frac{5}{2}$ and $y = 5$.

31. In the previous problem, show that for the value of x minimizing the length of the cable, the three angles emanating from the middle point P are all equal to 120°. Assume that $D \le 2\sqrt{3}\,L$ so that $x = c$. *Hint:* let $\theta = \angle APB$. By symmetry, it suffices to show that the upper angle $\theta = 120°$. To do this, observe that $D/2x = \tan\theta/2$.

33. In the set-up of the previous problem, how often should a truck be purchased so as to minimize average yearly cost assuming that after t years the old truck can be sold for $28000(1+\sqrt{t})^{-1}$ dollars?

A new truck costs \$30,000, the cost of maintaining it t years is $C(t) = 1280 + 360\left(t^2 + t\right)$ dollars, and after t years the old truck can be sold for $\dfrac{28000}{1+\sqrt{t}}$ dollars.

(a) The total cost for keeping a truck t years is $30000 + C(t) - \dfrac{28000}{1+\sqrt{t}}$. Hence the average cost of keeping it t years is

$$A(t) = \frac{1}{t}\left(30000 + C(t) - \frac{28000}{1+\sqrt{t}}\right) = 360t + 360 + 31280t^{-1} - \frac{28000}{t\left(1+\sqrt{t}\right)}$$

(b) Numerically solve

$$A'(t) = 360 - 31280t^{-2} + \frac{28000}{t^2\left(1+\sqrt{t}\right)} + \frac{14000}{t^{3/2}\left(1+\sqrt{t}\right)^2} = 0$$

for $t > 0$ to obtain $t \approx 7.659$ years. Now $A(t) \uparrow \infty$ as $t \downarrow 0$ or $t \uparrow \infty$. Accordingly, the minimum average yearly cost is approximately $A(7.659) \approx \$6{,}230.96$ when $t \approx 7.659$ years. A new truck should be purchased every 7.659 years to minimize costs.

The next two exercises refer to Example **??** in the text.

35. Suppose that the points A and B have the same distance from the mirror (i.e., $h_1 = h_2$).

(a) Show that $\theta_1 = \theta_2$.

(b) Can you find a reason why we must have $\theta_1 = \theta_2$ in this case, without doing any calculation? *Hint:* use symmetry.

Suppose that A and B have the same distance from the mirror; i.e., $h_1 = h_2 = h$.

(a) Substitute $h_1 = h_2 = h$ into

$$\frac{x}{\sqrt{x^2 + h_1^2}} = \frac{L - x}{\sqrt{(L-x)^2 + h_2^2}} \quad \text{to obtain} \quad \frac{x}{\sqrt{x^2 + h^2}} = \frac{L - x}{\sqrt{(L-x)^2 + h^2}}.$$

Solving for x gives $x = \frac{L}{2}$. Similar triangles then gives $\theta_1 = \theta_2$.

(b) Since $h_1 = h_2 = h$, the diagram must look the same from the front and back of the page. This can only occur if $\theta_1 = \theta_2$.

In the next three problems (courtesy of Kay Dundas), a pizza box (not including its top) of height h is to be constructed from a piece of cardboard of dimensions $A \times B$ (see Figure 9).

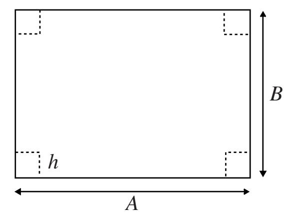


Figure 9

37. **(a)** Show that the volume of the box constructed from cardboard as in Figure 9 is

$$V(h) = h(A - 2h)(B - 2h)$$

(b) Which value of h maximizes the volume if $A = 15$ and $B = 24$? What are the dimensions of the resulting box?

(a) Use the diagram in Exercise **??** (except substitute h for t) with $x + 2h = A = 15$ and $y + 2h = B = 24$, whence $x = A - 2h = 15 - 2h$ and $y = B - 2h = 24 - 2h$. The volume of the box is
$V(h) = hxy = t(A - 2h)(B - 2h) = t(15 - 2h)(24 - 2h) = 4h^3 - 78h^2 + 360h$, where $0 \le h \le \frac{15}{2}$ (allowing for degenerate boxes). Note that $V(0) = V\left(\frac{15}{2}\right) = 0$, as expected.

(b) Solve $V'(h) = 12h^2 - 156h + 360 = 0$ for h to obtain $h = 3$ or $h = 10$. Toss out $h = 10$ (since it's outside the range of h). This leaves height $h = 3$ with maximum volume $V(3) = 486$ from a box of dimensions $h = 3$, $x = 9$, and $y = 18$.

39. Suppose that 144 square inches of cardboard is to be used to construct the pizza box ($AB = 144$).

(a) Find the values of A and B that give a box of maximum volume, assuming the height of the box is $h = 3$.

(b) Suppose more generally that $h = c$ where c is a constant. Show that the values of A and B giving a box of maximum volume does not depend on c.

(a) Use the diagram in Exercise **??** with $t = 3$, $x + 2t = A$, $y + 2t = B$, and $AB = 144$. From the four preceding equations we have $(x + 6)(y + 6) = 144$, whence
$y = \dfrac{144}{x + 6} - 6 = \dfrac{108 - 6x}{x + 6} = \dfrac{6(18 - x)}{x + 6}$.

- The volume of the box is $V(x) = txy = \dfrac{18x(18 - x)}{x + 6}$, where $0 \le x \le 18$ (allowing for degenerate boxes). Note that $V(0) = V(18) = 0$ as expected.
- Solve $V'(x) = \dfrac{-18\left(x^2 + 12x - 18\right)}{(x + 6)^2} = 0$ to obtain $x = -18$ (outside the range of x) and $x = 6$ at which the maximum volume $V(6) = 108$ occurs. Thus $y = 6$, $A = 12$, and $B = 12$, with all lengths in inches.

(b) Use the diagram in Exercise **??** with $t = c > 0$, $x + 2t = A$, $y + 2t = B$, and $AB = 144$. From the four preceding equations we have $(x + 2c)(y + 2c) = 144$, whence $y = \dfrac{2\left(72 - 2c^2 - cx\right)}{x + 2c}$.

- The volume of the box is $V(x) = txy = \dfrac{2cx\left(72 - 2c^2 - cx\right)}{x + 2c}$, where $0 \le x \le \dfrac{72}{c} - 2c$ (allowing for degenerate boxes). Note that $V(0) = V\left(\dfrac{72}{c} - 2c\right) = 0$ as expected.
- Solve $V'(x) = \dfrac{-2c^2\left(x^2 + 4cx + 4c^2 - 144\right)}{(x + 2c)^2} = 0$ to obtain $x = -2c - 12$ (outside the range of x) and $x = 12 - 2c$ at which the maximum volume $V(12 - 2c) = 4c(6 - c)^2$ occurs. Thus $y = 12 - 2c$, $A = 12$, and $B = 12$, with all lengths in inches.

41. (a) Show that among all right triangles with hypotenuse of length one, the isosceles triangle has maximum area.

(b) Can you see more directly why this must be true by reasoning from Figure 16?

(a) Position the right angle of the triangle at the origin of the xy-plane with its legs along the positive x- and y-axes. Since the hypotenuse has length 1, the area of the triangle is $A(x) = \frac{1}{2}xy = \frac{1}{2}x\sqrt{1 - x^2}$, $0 \le x \le 1$, allowing for degenerate triangles. Solve $A'(x) = \dfrac{\sqrt{1 - x^2}}{2} - \dfrac{x^2}{2\sqrt{1 - x^2}} = 0$ for $0 \le x \le 1$ to obtain $x = \dfrac{\sqrt{2}}{2}$. Since $A(0) = A(1) = 0$, the maximum area of $A(\frac{\sqrt{2}}{2}) = \frac{1}{4}$ occurs when $x = \frac{\sqrt{2}}{2}$. The sine of angle θ between the positive y-axis and the hypotenuse is $\sin\theta = x/1 = \frac{\sqrt{2}}{2}$, whence $\theta = \frac{\pi}{4}$. Therefore, the triangle isosceles.

(b) In the figure, let θ be the acute angle above the label a. Let $f(\theta)$ be the area of the right triangle whose hypotenuse is the diameter of the semicircle. Allowing for degenerate triangles, we have $0 \le \theta \le \frac{\pi}{2}$. Since f is symmetric with respect to the value $\theta = \frac{\pi}{4}$ (i.e., for $0 \le \theta \le \frac{\pi}{4}$, we have $f\left(\frac{\pi}{2} - \theta\right) = f(\theta)$) and because $f(0) = f\left(\frac{\pi}{2}\right) = 0$, the maximum area must occur when $\theta = \frac{\pi}{4}$. In this case the triangle in question is isosceles.

43. In the setting of Exercise 42, show that the minimal force required is proportional to $1/\sqrt{1 + f^2}$. *Hint:* find the optimal angle α and then compute $F(\alpha)$.

Let $F(\alpha) = \dfrac{k}{\sin\alpha + f\cos\alpha}$, where $k > 0$ is a proportionality constant and $0 \le \alpha \le \frac{\pi}{2}$. Solve

$$F'(\alpha) = \frac{k\,(f\sin\alpha - \cos\alpha)}{\left(\sin\alpha + \frac{2}{5}\cos\alpha\right)^2} = 0$$

for $0 \le \alpha \le \frac{\pi}{2}$ to obtain $\alpha = \tan^{-1}(\frac{1}{f})$. Since $F(0) = \frac{1}{f}k$ and $F(\frac{\pi}{2}) = k$, we conclude that the minimum force of $F(\tan^{-1}(\dfrac{1}{f})) = \dfrac{k}{\sqrt{1 + f^2}}$ is required when $\alpha = \tan^{-1}(\frac{1}{f})$.

Further Insights and Challenges

45. R & W Continuing with the notation of the previous exercise, aerodynamic analysis shows that in general, the power (in J/s) consumed by a bird flying at velocity v (in m/s) is described by a function $P(v) = av^{-1} + bv^3$ where a, b are positive constants that depend on certain characteristics of the bird. The total distance it can travel at velocity v is $D(v) = Ev/P(v)$ where E (a constant) is the usable energy it can store as body fat. Show that

$$v_{\text{dmax}} > v_{\text{pmin}}$$

where $v_{\text{dmax}}, v_{\text{pmin}}$ are as in the previous exercise.

Let $P(v) = av^{-1} + bv^3$. Then $P'(v) = -av^{-2} + 3bv^2 = 0$ implies $v_{\text{pmin}} = \left(\frac{a}{3b}\right)^{1/4}$. To find v_{dmax}, use $P'(v) = \dfrac{P(v)}{v}$. This gives $-av^{-2} + 3bv^2 = \dfrac{av^{-1} + bv^3}{v} = av^{-2} + bv^2$ which, simplified, yields $2bv^4 = 2a$ and finally $v_{\text{dmax}} = \left(\frac{a}{b}\right)^{1/4}$. Then $v_{\text{pmin}} = \left(\frac{a}{3b}\right)^{1/4} < \left(\frac{a}{b}\right)^{1/4} = v_{\text{dmax}}$. Notice the difference between the two values is the 3 in the denominator of v_{pmin}.

47. Gasoline is delivered to a gas station every t days. Suppose that the cost per delivery is d dollars and the average daily cost of storing the gasoline is st dollars where d and s are constants. Show that $f(t) = dt^{-1} + st$ represents the total average daily cost and find the value of t that minimizes average daily cost.

The total cost to operate the gas station over t days is $d + t \cdot st$. Thus the total average daily cost is $f(t) = \frac{d+t\cdot st}{t} = \frac{d}{t} + st$. Solve $f'(t) = s - dt^{-2} = 0$ for $t > 0$ to obtain $t = \sqrt{\frac{d}{s}}$. Since $f''(t) = 2d/t^3 > 0$ for all $t > 0$, the minimum average daily cost is $f(\sqrt{\frac{d}{s}}) = 2\sqrt{ds}$ when $t = \sqrt{\frac{d}{s}}$.

49. **(a)** Find the radius and height of a cylindrical can of total surface area A whose volume is as large as possible.
(b) Can you design a cylinder with total surface area A and minimal total volume?

Let a closed cylindrical can be of radius r and height h.

(a) Its total surface area is $S = 2\pi r^2 + 2\pi rh = A$, whence $h = \dfrac{A}{2\pi r} - r$. Its volume is thus $V(r) = \pi r^2 h = \frac{1}{2}Ar - \pi r^3$. Solve $V'(r) = \frac{1}{2}A - 3\pi r^2$ for $r > 0$ to obtain $r = \sqrt{\dfrac{A}{6\pi}}$. Since $V''(r) = -6\pi r < 0$ for $r > 0$, the maximum volume is

$$V\left(\sqrt{\frac{A}{6\pi}}\right) = \frac{\sqrt{6}A^{3/2}}{18\sqrt{\pi}} \quad \text{when} \quad r = \sqrt{\frac{A}{6\pi}} \quad \text{and} \quad h = \frac{1}{3}\sqrt{\frac{6A}{\pi}}.$$

(b) For a can of total surface area A, there are cans of arbitrarily small volume since $\lim_{r\to 0+} V(r) = 0$.

51. **Snell's Law** The x-axis represents the surface of a swimming pool. A light beam travels from point A located above the pool to point B located underneath the water. Let v_1 be the velocity of light in air and let v_2 be the velocity of light in water (it is a fact that $v_2 < v_1$). Show that the path from A to B that takes the *least time* satisfies the relation

$$\frac{\sin(\theta_1)}{v_1} = \frac{\sin(\theta_2)}{v_2}.$$

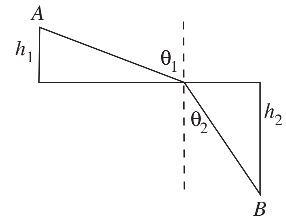
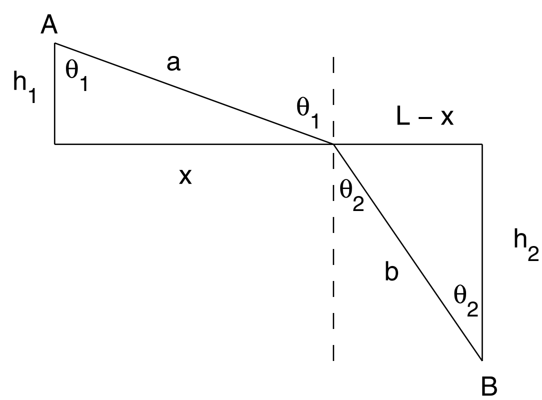


Figure 12

This relation is called Snell's Law of Refraction.

The time it takes a beam of light to travel from A to B is
$f(x) = \dfrac{a}{v_1} + \dfrac{b}{v_2} = \dfrac{\sqrt{x^2+h_1^2}}{v_1} + \dfrac{\sqrt{(L-x)^2+h_2^2}}{v_2}$. (See diagram below.) Now
$f'(x) = \dfrac{x}{v_1\sqrt{x^2+h_1^2}} - \dfrac{L-x}{v_2\sqrt{(L-x)^2+h_2^2}} = 0$ yields
$\dfrac{x\Big/\sqrt{x^2+h_1^2}}{v_1} = \dfrac{(L-x)\Big/\sqrt{(L-x)^2+h_2^2}}{v_2}$ or $\dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$, which is Snell's Law.
Since $f''(x) = \dfrac{h_1^2}{v_1\left(x^2+h_1^2\right)^{3/2}} + \dfrac{h_2^2}{v_2\left((L-x)^2+h_2^2\right)^{3/2}} > 0$ for all x, the minimum time is realized when Snell's Law is satisfied.



53. The Interval Matters This exercise analyzes the solution to Example **??** in a more general setting. Suppose that v_d and v_h are the speed limits on the dirt road and highway, respectively. Let ℓ be the distance from the ranch to the highway. Let P be the point on the highway nearest the ranch and d the distance from P to the city. In the example, $v_d = 20$, $v_h = 55$, $\ell = 4$ and $d = 9$. Let x_0 be the optimal value of x minimizing the time of the trip.

(a) Over which closed interval is the optimization taking place?

(b) Find a formula $f(x)$ for the time of the trip (in terms of v_d, v_h, ℓ, and d) and prove that $x_0 = \ell v_d/\sqrt{v_h^2 - v_d^2}$.

(c) This formula does not make sense if $v_h < v_d$. Why? Where does the minimum of $f(x)$ in $[0, d]$ occur in this case?

(d) Observe that the formula for x_0 must also be wrong if $\ell > \sqrt{v_h^2 - v_d^2}\,d/v_d$ since this would give us $x_0 > d$. Explain why $f(x)$ no longer represents the time of the trip when $x > d$. What is x_0 in this case?

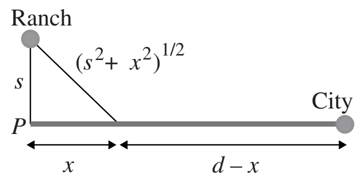


Figure 13

(a) The interval of optimization is $0 \le x \le d$.

(b) The time of the trip is $f(x) = \dfrac{\sqrt{s^2+x^2}}{v_d} + \dfrac{d-x}{v_h}$. Solve $f'(x) = \dfrac{x}{v_d\sqrt{s^2+x^2}} - \dfrac{1}{v_h} = 0$ for x in the stated range to obtain $x = x_0 = \dfrac{s v_d}{\sqrt{v_h^2 - v_d^2}}$. Since $f''(x) = \dfrac{s^2}{v_d\,(s^2+x^2)^{3/2}} > 0$ for all x, we have that the minimum time for the trip is given by $f(x_0)$ where $x_0 = \dfrac{s v_d}{\sqrt{v_h^2 - v_d^2}}$.

(c) If $v_h < v_d$ in the formula for x_0, then x_0 is complex-valued, which is nonsensical. In this case, $f(d)$ gives the minimum time. In other words, drive in a straight line between the ranch and the city on the dirt road to reach the destination as fast as possible (since traveling over the dirt road is faster than traveling on the highway in this case).

(d) In the event that $s > \dfrac{d\sqrt{v_h^2 - v_d^2}}{v_d}$, we then have $x_0 > d$. But $f(x_0)$ no longer gives the minimal time, since $x_0 > d$ means that the driver has to backtrack to the city along the highway. In this case, $x_0 = 0$ results in the minimal total time.

55. The problem is to put a "roof" of side s on a rectangle of height h and base b. Find the smallest length s for which this is possible.

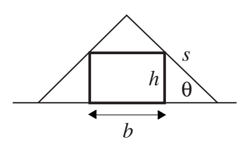


Figure 14

Place the base of the rectangle along the x-axis in the xy-plane with its center at the origin. Let $a = b/2$ be half the length of the base. (This gives a cleaner formulation in what is to follow.) Let P be the point at the peak of the roof, Q be the point where the "right" side of the roof (in the first quadrant) intersects the x-axis, and L the *full* length of the line segment from P to Q; i.e., the slanted length of the roof on the right side. We'll minimize the *square* of this length L, an equivalent problem with simpler (hand) computations.

- The right side of the roof touches the upper right corner of the rectangle at $R\,(a, h)$, a point in the first quadrant (i.e., $a, h > 0$). The line through P and Q is $y = h + m(x - a)$, where m is its slope. Here m is negative: $-\infty < m < 0$.

- Hence P has coordinates $(0, h - am)$ and Q has coordinates $\left(a - \frac{h}{m}, 0\right)$. Via the distance formula, we have that the square of the distance L is
$$f(m) = \left(a - \frac{h}{m}\right)^2 + (h - am)^2 = \frac{(am - h)^2\left(m^2 + 1\right)}{m^2}.$$
- Solve $f'(m) = \dfrac{2(ma - h)\left(m^3 a + h\right)}{m^3} = 0$ for $m < 0$ to obtain $m = m_0 = -\dfrac{h^{1/3}}{a^{1/3}}$. Since $f \uparrow \infty$ as $m \downarrow -\infty$ or $m \uparrow 0$, the minimum value of f is $f\left(-\dfrac{h^{1/3}}{a^{1/3}}\right) = \left(a^{2/3} + h^{2/3}\right)^3$, the minimum value of L^2, the square of the length L.
- Accordingly the minimum value of L is $L_0 = \left(a^{2/3} + h^{2/3}\right)^{3/2} = \left((b/2)^{2/3} + h^{2/3}\right)^{3/2}$.
- For $m = m_0 = -\dfrac{h^{1/3}}{a^{1/3}}$, the coordinates of Q are $\left(a + a^{1/3}h^{2/3}, 0\right)$. The distance between Q and R is
$$s = h^{2/3}\sqrt{a^{2/3} + h^{2/3}} \quad \text{or} \quad s = h^{2/3}\sqrt{(b/2)^{2/3} + h^{2/3}}.$$

57. (Adapted from A Problem in Maxima and Minima, Roger Johnson, *Am. Math. Monthly* **35** (1928), pp. 187–188.) A jewelry designer plans to incorporate a component made of gold in the shape of a frustum of a cone of height 1 cm and fixed lower radius r. The upper radius x can take on any value between 0 and r. Note that $x = 0$ and $x = r$ correspond to a cone and cylinder, respectively. As a function of x, the surface area (not including the top and bottom) is $S(x) = \pi(r + x)s$ where s is the *slant height* as indicated in Figure 15. Which value of x yields the least expensive design (the minimum value of $S(x)$ for $0 \le x \le r$)?

(a) Show that $s = \sqrt{1 + (r - x)^2}$, and hence $S(x) = \pi(r + x)\sqrt{1 + (r - x)^2}$.

(b) Show that if $r < \sqrt{2}$, then $S(x)$ is an increasing function. Conclude that the cone ($x = 0$) has minimal area in this case.

(c) Assume that $r > \sqrt{2}$. Show that $S(x)$ has two critical points $x_1 < x_2$ in $(0, r)$ and that $S(x_1)$ is a local maximum, $S(x_2)$ is a local minimum.

(d) Conclude that the minimum value occurs either at $x = 0$ or at $x = x_2$.

(e) Where does the minimum value occur in the cases $r = 1.5$ and $r = 2$?

(f) Challenge: let $c = \sqrt{(5 + 3\sqrt{3})/4} \approx 1.5966$. Prove that the minimum occurs at $x = 0$ (cylinder) if $\sqrt{2} < r < c$ but the minimum occurs at $x = x_2$ if $r > c$.

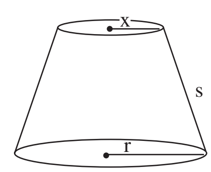


Figure 15

(a) Consider a cross-section of the object and notice a triangle can be formed with height 1, hypotenuse s, and base $r - x$. Then, by the Pythagorean Theorem, $s = \sqrt{1 + (r - x)^2}$ and the surface area $S = \pi(r + x)s = \pi(r + x)\sqrt{1 + (r - x)^2}$.

(b) $S'(x) = \pi\left(\sqrt{1 + (r - x)^2} - (r + x)(1 + (r - x)^2)^{-1/2}(r - x)\right) = \pi\dfrac{2x^2 - 2rx + 1}{\sqrt{1 + (r - x)^2}} = 0$ yields critical points $x = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - 2}$. If $r < \sqrt{2}$ then there are no real critical points. Sign analyses reveal that $S(x)$ is increasing everywhere and thus the minimum must occur at the left endpoint, $x = 0$.

(c) For $r < \sqrt{2}$, there are two critical points, $x = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 - 2}$. Both values are in the interval since $r > \sqrt{r^2 - 2}$. Sign analyses reveal that $x = \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2}$ is a local minimum and $x = \frac{1}{2}r - \frac{1}{2}\sqrt{r^2 - 2}$ is a local maximum.

(d) Depending on the value of r, the minimum of $S(x)$ occurs at either $x = 0$ or $x = \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2}$.

(e) 1.5 and 2 are both greater than $\sqrt{2}$. This means that, for both values of r, the local minimum occurs at $x = \frac{1}{2}r + \frac{1}{2}\sqrt{r^2 - 2}$. If $r = 1.5$, $S(x_2) = 8.8357$ and $S(0) = 8.4954$, so $S(0) = 8.4954$ is the local minimum (cylinder). If $r = 2$, $S(x_2) = 12.852$ and $S(0) = 14.0496$, so $S(x_2) = 12.852$ is the minimum.

(f)

Seismic Prospecting (Adapted from B. Noble, *Applications of Undergraduate Mathematics in Engineering*, Macmillan 1967.) Exercises 59–61 are concerned with determining the thickness d of a layer of soil that lies on top of a rock formation. Geologists send two sound pulses from point A to point D separated by a distance s. The first pulse travels directly from A to D along the surface of the earth. The second pulse travels down to the rock formation, then along its surface, and then back up to D (path $ABCD$) as in Figure 17. The pulse travels with velocity v_1 in the soil and v_2 in the rock.

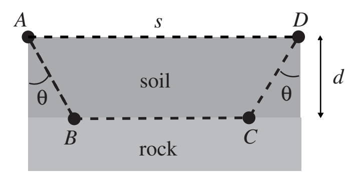


Figure 17

59. **(a)** Show that the time required for the first pulse to travel from A to D is $t_1 = s/v_1$.

(b) Show that the time required for the second pulse is

$$t_2 = \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2}$$

provided that

$$\tan\theta \le \frac{s}{2d} \tag{3}$$

(Note: if this inequality is not satisfied, then point B does not lie to the left of C.)

(c) Show that t_2 is minimized when $\sin\theta = v_1/v_2$.

Seismic Prospecting.

(a) We have time $t_1 = \text{distance}/\text{velocity} = s/v_1$.

(b) Let p be the length of the base of the right triangle (opposite the angle θ) and h the length of the hypotenuse of this right triangle. Then $\cos\theta = \frac{d}{h}$ whence $h = d\sec\theta$. Moreover, $\tan\theta = \frac{p}{d}$ gives $p = d\tan\theta$. Hence

$$t_2 = t_{AB} + t_{CD} + t_{BC} = \frac{h}{v_1} + \frac{h}{v_1} + \frac{s - 2p}{v_2} = \frac{2d}{v_1}\sec\theta + \frac{s - 2d\tan\theta}{v_2}$$

(c) Solve $\dfrac{dt_2}{d\theta} = \dfrac{2d\sec\theta\tan\theta}{v_1} - \dfrac{2d\sec^2\theta}{v_2} = 0$ to obtain $\dfrac{\tan\theta}{v_1} = \dfrac{\sec\theta}{v_2}$ whence $\dfrac{\sin\theta/\cos\theta}{1/\cos\theta} = \dfrac{v_1}{v_2}$ or $\sin\theta = \dfrac{v_1}{v_2}$.

61. Continue with the assumption of the previous exercise.

(a) Find the thickness of the top layer, assuming that $v_1 = .7v_2$ and that $t_2/t_1 = 1.3$ when A and B are separated by a distance $s = 1500$ ft.

(b) The times t_1, t_2 and the distance s are measured experimentally. The equation in Exercise 60(c) shows that t_2/t_1 is a linear function of $1/s$. What might you conclude if several experiments were formed for different values of s and the plot of the points $(1/s, t_2/t_1)$ did *not* lie on a straight line?

(a) Substituting $k = v_1/v_2 = 0.7$, $t_2/t_1 = 1.3$, and $s = 1500$ into (*) gives $1.3 = \dfrac{2d\sqrt{1-(0.7)^2}}{1500} + 0.7$. Solving for d yields $d \approx 630.13$ ft.

(b) If several experiments for different values of s showed that plots of the points $\left(\dfrac{1}{s}, \dfrac{t_2}{t_1}\right)$ did *not* lie on a straight line, then we would suspect that $\dfrac{t_2}{t_1}$ is *not* a linear function of $\dfrac{1}{s}$ and that a different model is required.

4.7 Newton's Method

Preliminary Questions

1. How many iterations of Newton's Method are required to compute a root if $f(x)$ is a linear function?

2. What happens if you start Newton's method with an initial guess that happens to be a zero of f?

3. What happens if you start Newton's method with an initial guess that happens to be a local minimum or maximum of f?

4. For which of the following two curves will Newton's Method converge more rapidly?

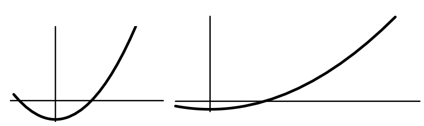


Figure 1

5. Is the statement "A root of the equation of the tangent line to $f(x)$ is used as an approximation to a root of $f(x)$ itself" a reasonable description of Newton's method? Explain.

Exercises

1. Use Newton's Method to calculate the approximations x_1, x_2, x_3 to the root of the equation $f(x) = x^2 - 10$. Take $x_0 = 3$.

Let $f(x) = x^2 - 10$. Define $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 10}{2x}$. Take $x_0 = 3$, and $x_{n+1} = g(x_n)$. We get the following table:

n	1	2	3	4
x_n	3.166666666	3.162280701	3.162277660	3.162277660

3. Let $f(x) = x^3 - 7x + 3$. Find values a and b such that $f(a) > 0$ and $f(b) < 0$ by trial and error. Then use Newton's Method to find the root of f lying between a and b to three decimal places.

Let $f(x) = x^3 - 7x + 3$. Define $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 - 7x + 3}{3x^2 - 7}$.

- Taking $x_0 = -3$, we have

n	1	2	3	4
x_n	-2.85	-2.838534619	-2.838469254	-2.838469252

- Taking $x_0 = 0$, we have

n	1	2	3	4
x_n	0.428571429	0.440777577	0.440807711	0.440807712

- Taking $x_0 = 2$, we have

n	1	2	3	4	5
x_n	2.6	2.421084337	2.398036788	2.39766164	2.39766154

In Exercises 5–6, calculate x_1, x_2, x_3 using Newton's Method for the given function and initial guess. Compare x_3 with the value obtained from a calculator.

5. $f(x) = x^2 - 7, \quad x_0 = 2.5$

Let $f(x) = x^2 - 7$. Define $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - 7}{2x}$. Take $x_0 = 2.5$.

n	1	2	3	4
x_n	2.65	2.645754717	2.645751311	2.645751311

7. Choose an initial guess x_0 and calculate x_1, x_2, x_3 using Newton's Method to approximate a solution to the equation $e^x = 3x + 2$.

Let $f(x) = e^x - 3x - 2$. Let $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{e^x - 3x - 2}{e^x - 3}$. Let $x_0 = 0$.

n	1	2	3	4
x_n	-0.5	-.455491111	-.455233364	-.455233355

9. Let $f(x) = x^4 - 6x^2 + x + 5$ as in Example **??**. Use Newton's Method to approximate the largest positive root of $f(x)$ to within an error of at most 10^{-4}. (Refer to Figure **??**.)

The largest positive root of $f(x)$ is $x = 2.09306435$. Let $f(x) = x^4 - 6x^2 + x + 5$. Define

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^4 - 6x^2 + x + 5}{4x^3 - 12x + 1}.$$

Take $x_0 = 2$.

n	1	2	3
x_n	2.11111111	2.09356846	2.09306477

11. Use Figure 2 to choose an initial guess x_0 to the unique real root of $x^3 + 2x + 5 = 0$. Then compute the first 3 iterates of Newton's Method.

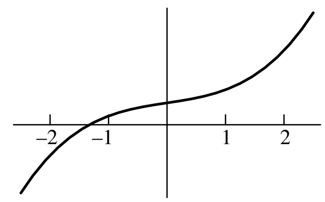


Figure 2 Graph of $y = x^3 + 2x + 5$.

Let $f(x) = x^3 + 2x + 5$. Define $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^3 + 2x + 5}{3x^2 + 2}$. Take $x_0 = -1.4$, based on the figure.

n	1	2	3
x_n	−1.330964467	−1.328272820	−1.328268856

13. GU Use a graphing calculator to choose an initial guess for the unique positive root of $x^4 + x^2 - 2x - 1 = 0$. Calculate the first three iterates of Newton's Method.

Let $f(x) = x^4 + x^2 - 2x - 1$. Define $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^4 + x^2 - 2x - 1}{4x^3 + 2x - 2}$. Take $x_0 = 1$.

n	1	2	3
x_n	1.25	1.189379699	1.18417128

15. Show that the sequence converging to $\sqrt{c}$ obtained by applying Newton's Method is defined by $x_{n+1} = \frac{1}{2}(x_n + \frac{c}{x_n})$.

Let $f(x) = x^2 - c$. Define $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^2 - c}{2x} = \frac{x^2 + c}{2x} = \frac{1}{2}\left(x + \frac{c}{x}\right)$.
Then $x_{n+1} = \frac{1}{2}\left(x_n + \frac{c}{x_n}\right)$.

17. What happens when you apply Newton's Method to find a zero of $f(x) = x^{1/3}$? Note that $x = 0$ is the only zero.

Let $f(x) = x^{1/3}$. Define $g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{x^{1/3}}{\frac{1}{3}x^{-2/3}} = x - 3x = -2x$. Take $x_0 = 0.5$. Then the sequence of iterates is $-1, 2, -4, 8, -16, 32, -64, \ldots$ That is, for any nonzero starting value, the sequence of iterates diverges spectacularly, since $x_n = (-2)^n x_0$. Thus $\lim_{n\to\infty} |x_n| = \lim_{n\to\infty} 2^n |x_0| = \infty$.

Further Insights and Challenges

19. Let c be a positive number and let $f(x) = x^{-1} - c$.

(a) Show that $x - (f(x)/f'(x)) = 2x - cx^2$. Thus Newton's Method provides a way of computing reciprocals without performing division.

(b) Calculate the first three iterates of Newton's Method with $c = 10.324$ and the two initial guesses $x_0 = .1$ and $x_0 = .5$.

(c) Explain graphically why $x_0 = .5$ does not yield a sequence of approximations to the reciprocal $1/10.324$.

(a) Let $f(x) = \frac{1}{x} - c$. Define $g(x) = x - \dfrac{f(x)}{f'(x)} = x - \dfrac{\frac{1}{x} - c}{-x^{-2}} = 2x - cx^2$.

(b) For $c = 10.324$, we have $f(x) = \frac{1}{x} - 10.324$ and thus $g(x) = 2x - 10.324x^2$.

- Take $x_0 = 0.1$.

n	1	2	3
x_n	0.09676	0.096861575	0.096861682

- Take $x_0 = 0.5$.

n	1	2	3
x_n	-1.581	-28.9674677	-8.72094982

(c) The graph is disconnected. If $x_0 = .5$, $(x_1, f(x_1))$ is on the other portion of the graph, which will never converge to any point under Newton's method.

21. Let $f(\theta) = \dfrac{\sin\theta}{\theta}$.

(a) Using a graphing calculator or other method, choose an initial guess for a solution to $f(\theta) = .9$.

(b) Use Newton's Method to improve the guess to three decimal accuracy.

Let $f(x) = \dfrac{\sin x}{x}$ and let $0 < \alpha < 1$.

(a) Let $h(x) = \dfrac{\sin x}{x} - \alpha$. Define

$$g(x) = x - \frac{h(x)}{h'(x)} = x - \frac{\frac{\sin x}{x} - \alpha}{\frac{x\cos x - \sin x \cdot 1}{x^2}} = \frac{x\,(x\cos x - 2\sin x + \alpha x)}{x\cos x - \sin x}.$$

The sequence of iterates is given by $x_{n+1} = g(x_n)$.

(b) For $x_0 = 0.8$, we have $f(x_0) - \alpha = \frac{\sin 0.8}{0.8} - 0.9 \approx -0.0033$. So let's use $x_0 = 0.8$ for our initial guess.

(c) Then

n	1	2	3
x_n	0.786779688	0.786683077	0.786683072

In the next two exercises (adapted from *Animating Calculus*, Packel and Wagon, p. 79), we consider a metal rod of length L inches that is fastened at both ends. If you cut the rod and weld on an additional m inches of rod leaving the ends fixed, the rod will bow up into a circular arc of radius R (unknown) as indicated in Figure 5.

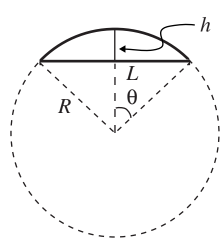


Figure 5 The bold circular arc has length $L + m$.

23. Let $L = 1$ and $m = 3$.

(a) Use Newton's Method to find the angle θ satisfying Eq. (1) to within an error of 10^{-3}. Use a graphing calculator to make an initial guess.

(b) Estimate the value of h.

Let $L = 1$ and $m = 3$.

(a) From Exercise **??**(b), we deduce that $\frac{\sin\theta}{\theta} - \frac{L}{L+m} = 0$ or $\frac{\sin\theta}{\theta} - \frac{1}{4} = 0$. Applying Exercise 21 with $\alpha = \frac{1}{4}$ produces the following Newton iterates, which converge to θ.

n	0	1	2	3	4
θ_n	2	2.470027008	2.474574622	2.474576787	2.474576787

. That is, $\theta \approx 2.474567687$.

(b) Accordingly, $h = \dfrac{L(1-\cos\theta)}{2\sin\theta} \approx 1.443213471$.

4.8 Antiderivatives

Preliminary Questions

1. Find an antiderivative of the function $f(x) = 0$.

2. What is the difference, if any, between finding the general antiderivative of a function $f(x)$ and evaluating $\int f(x)\,dx$?

3. Joe happens to know that two functions $f(x)$ and $g(x)$ have the same derivative, and he would like to know if $f(x) = g(x)$. Does Joe have sufficient information to answer his question?

4. Is $y = x^2$ a solution to the differential equation with initial condition:

$$\frac{dy}{dx} = 2x, \qquad y(0) = 1$$

5. Write any two antiderivatives of $\cos x$. Which initial conditions do they satisfy at $x = 0$?

6. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Are the following statements true or false? Explain.

(a) If F and G differ by a constant, then $f = g$.

(b) If $F = G$, then $f = g$.

(c) If F and G differ by a constant, then f and g differ by a constant.

(d) If $f = g$, then $F = G$.

Exercises

1. Which of (A) or (B) is the graph of an antiderivative of $f(x)$ in Figure 1?

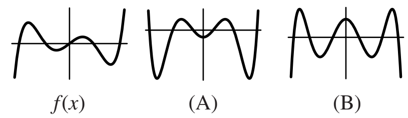


Figure 1

The graph of the antiderivative $F(x)$ of $f(x)$ is (A); because the graph of $f(x) = F'(x)$ is that of the slopes of $F(x)$.

Find the general antiderivative of the following functions in Exercises 3–6.

3. $12x$

$\int 12x\,dx = 12\int x\,dx = 12 \cdot \frac{1}{2}x^2 + C = 6x^2 + C.$

5. $x^2 + 3x + 2$

$\int x^2 + 3x + 2\,dx = \int x^2 dx + 3\int x^1\,dx + 2\int x^0\,dx = \frac{1}{3}x^3 + 3 \cdot \frac{1}{2}x^2 + 2 \cdot \frac{1}{1}x^1 + C = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x + C.$

7. Evaluate $\int (x^2 + 1)\,dx$. Check your answer by differentiation.

$\int x^2 + 1\,dx = \frac{1}{3}x^3 + x + C$ since
$\frac{d}{dx}\left(\frac{1}{3}x^3 + x + C\right) = \frac{d}{dx}\left(\frac{1}{3}x^3\right) + \frac{d}{dx}(x) + \frac{d}{dx}(C) = x^2 + 1 + 0 = x^2 + 1.$

9. Find the solution of the differential equation $dy/dx = x^3$ with initial condition $y(0) = 2$.

Since $\frac{dy}{dx} = x^3$, we have $y = \int x^3\,dx = \frac{1}{4}x^4 + C$. Thus $2 = y(0) = \frac{1}{4}(0)^4 + C$, whence $C = 2$. Therefore, $y = \frac{1}{4}x^4 + 2$. As a check, note that $\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{4}x^4 + 2\right) = x^3$ and $y(0) = \frac{1}{4}(0)^4 + 2 = 2$, as required.

11. A particle located at the origin at $t = 0$ begins moving along the x-axis with velocity $v(t) = \frac{1}{2}t^2 - t$ ft/s. Let $s(t)$ be the particle's position at time t. State the differential equation with initial condition satisfied by $s(t)$ and find $s(t)$.

Given that $v(t) = \frac{ds}{dt}$ and that the particle starts at the origin, the differential equation is $\frac{ds}{dt} = \frac{1}{2}t^2 - t$ and the initial condition is $s(0) = 0$. Accordingly, we have $s(t) = \int \frac{1}{2}t^2 - t\,dt = \frac{1}{2}\int t^2\,dt - \int t\,dt = \frac{1}{2} \cdot \frac{1}{3}t^3 - \frac{1}{2}t^2 + C = \frac{1}{6}t^3 - \frac{1}{2}t^2 + C$. Thus $0 = s(0) = \frac{1}{6}(0)^3 - \frac{1}{2}(0)^2 + C$, whence $C = 0$. Therefore, the position is $s(t) = \frac{1}{6}t^3 - \frac{1}{2}t^2$.

13. A particle located at the origin at $t = 0$ begins moving in a straight line with *acceleration* $a(t) = 4 - \frac{1}{2}t$ ft/s^2. Let $v(t)$ be the velocity and $s(t)$ be the distance traveled at time t.

(a) State and solve the differential equation assuming the particle is at rest at $t = 0$.

(b) Find $s(t)$.

(a) The differential equation is $a(t) = \frac{dv}{dt} = 4 - \frac{1}{2}t$ and its initial condition is $v(0) = v_0$ where v_0 is the (unspecified) initial velocity of the particle. We have $v(t) = \int 4 - \frac{1}{2}t\,dt = \int 4\,dt - \frac{1}{2}\int t\,dt = 4t - \frac{1}{4}t^2 + C$. Thus $v_0 = v(0) = 4(0) - \frac{1}{4}(0)^2 + C$, whence $C = v_0$. Therefore, the velocity is $v(t) = 4t - \frac{1}{4}t^2 + v_0$.

(b) In (a), we saw that $v(t) = \frac{ds}{dt} = 4t - \frac{1}{4}t^2 + v_0$. Accordingly,

$$s(t) = \int 4t - \frac{1}{4}t^2 + v_0\,dt = 4 \cdot \frac{1}{2}t^2 - \frac{1}{4} \cdot \frac{1}{3}t^3 + v_0 t + K = 2t^2 - \frac{1}{12}t^3 + v_0 t + K$$

Thus $8 = s(0) = 2\,(0)^2 - \frac{1}{12}\,(0)^2 + v_0\,(0) + K$, whence $K = 8$. Therefore, the position is $s(t) = 2t^2 - \frac{1}{12}t^3 + v_0 t + 8$.

In Exercises 15–30, evaluate the indefinite integral.

15. $\int (x+1)\,dx$

$\int x + 1\,dx = \frac{1}{2}x^2 + x + C.$

17. $\int (t^5 + 3t + 2)\,dt$

$\int t^5 + 3t + 2\,dt = \frac{1}{6}t^6 + \frac{3}{2}t^2 + 2t + C.$

19. $\int t^{-9/5}\,dt$

$\int t^{-9/5}\,dt = -\frac{5}{4}t^{-4/5} + C.$

21. $\int (5x^3 - x^{-2} - x^{3/5})\,dx$

$\int 5x^3 - x^{-2} - x^{3/5}\,dx = \frac{5}{4}x^4 + x^{-1} - \frac{5}{8}x^{8/5} + C.$

23. $\int \frac{1}{\sqrt{x}}\,dx$

$\int 1/\sqrt{x}\,dx = \int x^{-1/2}\,dx = 2x^{1/2} + C = 2\sqrt{x} + C.$

25. $\int (4\sin x - 3\cos x)\,dx$

$\int 4\sin x - 3\cos x\,dx = -4\cos x - 3\sin x + C.$

27. $\int (x^3 + 4x^{-2})\,dx$

$\int x^3 + 4x^{-2}\,dx = \frac{1}{4}x^4 - 4x^{-1} + C.$

29. $\int \sqrt{x}\,dx$

$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C.$

31. Use the formulas for the derivatives of $\tan x$ and $\sec x$ to evaluate the integrals

(a) $\int \sec^2(3x)\,dx$

(b) $\int \sec(x+3)\tan(x+3)\,dx$

Recall that $\frac{d}{dx}(\tan x) = \sec^2 x$ and $\frac{d}{dx}(\sec x) = \sec x \tan x$.

(a) Accordingly, we have $\int \sec^2(3x)\,dx = \frac{1}{3}\tan(3x) + C.$

(b) Moreover, we see that $\int \sec(x+3)\tan(x+3)\,dx = \sec(x+3) + C.$

33. A 900 kg rocket is fired from a spacecraft. As it burns fuel, its mass decreases and its velocity increases. Let v be the velocity as a function of mass m. Find the velocity when $m = 500$ if

$$dv/dm = -m^{-1/2}$$

Assume that $v(900) = 0$ (the velocity when first fired is 0).

Since $\frac{dv}{dm} = -m^{-1/2}$, we have $v(m) = \int -m^{-1/2}\,dm = -2m^{1/2} + C$. Thus $0 = v(900) = -2\sqrt{900} + C = -60 + C$, whence $C = 60$. Therefore, $v(m) = 60 - 2\sqrt{m}$. Accordingly, $v(500) = 60 - 2\sqrt{500} = 60 - 20\sqrt{5} \approx 15.28$.

35. **(a)** Show that the functions $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$ have the same derivative.

(b) What can you conclude about the relation between f and g? Verify this conclusion directly.

Let $f(x) = \tan^2 x$ and $g(x) = \sec^2 x$.

(a) Then $f'(x) = 2\tan x \sec^2 x$ and $g'(x) = 2\sec x \cdot \sec x \tan x = 2\tan x \sec^2 x$, whence $f'(x) = g'(x)$.

(b) Accordingly, the antiderivatives $f(x)$ and $g(x)$ must differ by a constant; i.e., $f(x) - g(x) = \tan^2 x - \sec^2 x = C$ for some constant C. To see that this is true directly, divide the identity $\sin^2 x + \cos^2 = 1$ by $\cos^2 x$. This yields $\tan^2 x + 1 = \sec^2 x$, whence $\tan^2 x - \sec^2 x = -1$.

In Exercises 36–45, solve the differential equation with initial condition.

37. $\dfrac{dy}{dt} = 0;\ \ y(3) = 5$

Since $\frac{dy}{dt} = 0$, we have $y = \int 0\,dt = C$. Thus $5 = y(3) = C$, whence $C = 5$. Therefore, $y \equiv 5$.

39. $\dfrac{dy}{dx} = 8x^3 + 3x^2 - 3;\ \ y(1) = 1$

Since $\frac{dy}{dx} = 8x^3 + 3x^2 - 3$, we have $y = \int 8x^3 + 3x^2 - 3\,dx = 2x^4 + x^3 - 3x + C$. Thus $1 = y(1) = 0 + C$, whence $C = 1$. Therefore, $y = 2x^4 + x^3 - 3x + 1$.

41. $\dfrac{dy}{dt} = \sqrt{t};\ \ y(1) = 1$

Since $\frac{dy}{dt} = \sqrt{t} = t^{1/2}$, we have $y = \int t^{1/2}\,dt = \frac{2}{3}t^{3/2} + C$. Thus $1 = y(1) = \frac{2}{3} + C$, whence $C = \frac{1}{3}$. Therefore, $y = \frac{2}{3}t^{3/2} + \frac{1}{3}$.

43. $\dfrac{dy}{dz} = \sin 2z;\ \ y\left(\dfrac{\pi}{4}\right) = 4$

Since $\frac{dy}{dz} = \sin 2z$, we have $y = \int \sin 2z\,dz = -\frac{1}{2}\cos 2z + C$. Thus $4 = y\left(\frac{\pi}{4}\right) = 0 + C$, whence $C = 4$. Therefore, $y = 4 - \frac{1}{2}\cos 2z$.

45. $\dfrac{dy}{dx} = \sec^2 3x;\ \ y\left(\dfrac{\pi}{4}\right) = 2$

Since $\frac{dy}{dx} = \sec^2 3x$, from Exercise 31(a) we have $y = \int \sec^2 3x\,dx = \frac{1}{3}\tan 3x + C$. Thus $2 = y\left(\frac{\pi}{4}\right) = -\frac{1}{3} + C$, whence $C = \frac{7}{3}$. Therefore, $y = \frac{7}{3} + \frac{1}{3}\tan 3x$.

47. Find the general antiderivative of $\cos 9t$.

$\int \cos 9t\, dt = \frac{1}{9}\sin 9t + C$.

49. Find the general antiderivative of $(2x+9)^{10}$.

Hint: try to guess the answer, based on the formula $\frac{d}{dx}(2x+9)^{11} = 22(2x+9)^{10}$.

By the Chain Rule, we have $d/dx\left((2x+9)^{11}\right) = 11\,(2x+9)^{10}\cdot 2 = 22\,(2x+9)^{10}$. Hence $\frac{d}{dx}\left(\frac{1}{22}\,(2x+9)^{11}\right) = (2x+9)^{10}$. Thus $\int (2x+9)^{10}\, dx = \frac{1}{22}\,(2x+9)^{11} + C$.

Further Insights and Challenges

51. Suppose that $F'(x) = f(x)$ and $G'(x) = g(x)$. Is it true that $F(x)G(x)$ an antiderivative of $f(x)g(x)$? Confirm or provide a counterexample.

Let $f(x) = x^2$ and $g(x) = x^3$. Then $F(x) = \frac{1}{3}x^3$ and $G(x) = \frac{1}{4}x^4$ are antiderivatives for $f(x)$ and $g(x)$, respectively. Let $h(x) = f(x)g(x) = x^5$, the general antiderivative of which is $H(x) = \frac{1}{6}x^6 + C$. There is no value of the constant C for which $F(x)G(x) = \frac{1}{12}x^7$ equals $H(x)$. Accordingly, $F(x)G(x)$ is *not* an antiderivative of $h(x) = f(x)g(x)$. Therefore, it is *not* true that $P'(x) = p(x)$ and $Q'(x) = q(x)$ imply that $P(x)Q(x)$ is an antiderivative of $p(x)q(x)$.

53. Find an antiderivative for $f(x) = |x|$.

Let $f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$. Then the general antiderivative of $f(x)$ is

$$F(x) = \int f(x)\, dx = \begin{cases} \int x\, dx & \text{for } x \geq 0 \\ \int -x\, dx & \text{for } x < 0 \end{cases} = \begin{cases} \frac{1}{2}x^2 + C & \text{for } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{for } x < 0 \end{cases}$$