7.4 p 452 #8

As \( x \to \infty \),

\[
\begin{align*}
(\ln 2)^x &< x^2 < e^x < 2^x \\
&\quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
&\quad \quad \quad c \quad \quad b \quad \quad a \quad \quad d
\end{align*}
\]

7.4 p 452 #10

As \( x \to \infty \),

a. \( \frac{1}{x+3} = \Theta\left(\frac{1}{x}\right) \) True
b. \( \frac{1}{x} + \frac{1}{x^2} = \Theta\left(\frac{1}{x}\right) \) True

c. \( \frac{1}{x} - \frac{1}{x^2} = o\left(\frac{1}{x}\right) \) False
d. \( 2 \ln x = \Theta(2) \) True
e. \( e^{x-x} = \Theta(e^x) \) True

f. \( x \ln x = o(x^2) \) True

g. \( \ln(x^2 + x) = \Theta(\ln x) \) True

(h. \( \ln x = o\left(\ln(x^2 + 1)\right) \) False

(in fact \( \ln(x^2 + 1) = o(\ln x) \))
10.1. p.581 # 5c
\( a_n = \frac{1}{n+1} \)

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n+1} = 0 \]

The sequence converges to 0.

10.1. p.581 # 48
\( a_n = \frac{1}{n} \)

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = 0 \]

The sequence converges to 0.

10.1. p.581 # 36
\( a_n = (-1)^n \)

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n \]

The sequence does not converge.

10.1. p.581 # 72
\( a_n = (1 - \frac{1}{n^2})^n \)

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} (1 - \frac{1}{n^2})^n = e^{-\frac{1}{2}} \]

The sequence converges to \( e^{-\frac{1}{2}} \).
10.1 p581 #94
\[ a_1 = 0, \quad a_{n+1} = \sqrt{8 + 2a_n} \]

Let \( \lim_{n \to \infty} a_n = L \) then

\[ L = \sqrt{8 + 2L} \Rightarrow L^2 = 8 + 2L \]

\[ L^2 - 2L - 8 = 0, \quad (L-4)(L+2)=0 \]

So either \( L = 4 \) or \( L = -2 \). But all the terms are positive so \( L \) can't be \(-2\). Thus \( \lim_{n \to \infty} a_n = 4 \).

10.2 p591 #24
\[ \sum_{n=1}^{\infty} \frac{414}{1000^n} = 1 + \frac{414}{1000} + \frac{414}{1000^2} + \cdots \]

geometric \( a = \frac{414}{1000} \), \( r = \frac{1}{1000} \)

\[ = 1 + \frac{414}{1000} \left( 1 - \frac{1}{1000} \right) = 1 + \frac{414}{999} \]

\[ = \frac{1413}{999} \]

10.2 p591 #30
\[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 3} \lim_{n \to \infty} \frac{n}{n^2 + 3} = 0, \text{ so } n^{\text{th}} \text{ term test is inconclusive.} \]

(Series diverges, though, by the integral test eg.)

10.2 p591 #34
\[ \sum_{n=0}^{\infty} (-1)^n \]

\[ = \text{sum of terms} \] (Series diverges by the nth term test)

10.2 p591 #38
\[ \sum_{n=1}^{\infty} (\tan^{-1} n - \tan^{-1} (n-1)) \]

\[ = (\tan^{-1} \infty) + (\tan^{-1} 2 - \tan^{-1} 1) + \cdots \]

The nth partial sum is \( \tan^{-1} \infty \); but \( \tan^{-1} \) does not have a limit as \( n \to \infty \), so the series diverges.

10.2 p591 #54
\[ \sum_{n=1}^{\infty} \frac{\cos n \pi}{5^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} \text{ geometric with } a = 1, \quad r = \frac{-1}{5} \]

Since \( |r| < 1 \), series converges to

\[ \frac{1}{1 - (-\frac{1}{5})} = \frac{5}{6} \]
See 10.2 p591 #60
\[ \sum_{n=1}^{\infty} \left( -\frac{1}{2} \right)^n \]
From #54 of Sec 10.1, we know \( \lim_{n \to \infty} \left( -\frac{1}{2} \right)^n = \frac{1}{2} \neq 0 \)
so series diverges by the \( n \)th term test.

See 10.2 p591 #76
\[ \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n (x-3)^n \]
Geometric with \( a=1 \) and \( r = -\frac{1}{2} (x-3) \).
Series will converge if \( |r| = \frac{1}{2} |x-3| < 1 \)
\[ 1-\frac{1}{2} |x-3| < 1 \]
i.e. \( |x-3| < 2 \) or \( 1 < x < 5 \)
Sum: \[ a \left( 1 - r \right) = \frac{1}{1 - \left( -\frac{1}{2} (x-3) \right)} = \frac{2}{5-x} \]

See 10.2 p591 #86
Since \( \sum \) an converges we know that \( a_n \to 0 \) as \( n \to \infty \). But then \( \frac{1}{a_n} \to \infty \) as \( n \to \infty \) so the series \( \sum \frac{1}{a_n} \) diverges by the \( n \)th term test.

See 10.2 p591 #96
A) Each time, every side of the figure is replaced by three sides, each of which has \( \frac{1}{3} \) the length of the original side. So \( L_1 = 3 \), \( L_2 = \frac{4}{3} \), \( L_3 = \frac{4}{3} \cdot \frac{4}{3} \), etc. \( L_n = \left( \frac{4}{3} \right)^n \) \( \to \infty \) as \( n \to \infty \).
Note: the number of sides of \( C_n \) is \( 3^n \).

See 10.3 p598 #4
\[ \sum_{n=1}^{\infty} \frac{1}{n+4} \]
Terms are positive and decreasing,
\[ \int_1^{\infty} \frac{1}{x+4} \, dx = \ln(x+4) \bigg|_1^{\infty} = \infty \] so series diverges.

See 10.3 p598 #6
\[ \sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} \]
Terms are positive and decreasing,
\[ \int_2^{\infty} \frac{1}{n \ln^2 n} \, dx = \int_2^{\infty} \frac{1}{u^2} \, du = -\frac{1}{u} \bigg|_2^{\ln 2} = \frac{1}{\ln 2} \]
\[ \Rightarrow \text{series converges} \]

See 10.3 p598 #16
\[ \sum_{n=1}^{\infty} \frac{2}{n^{3/2}} \]
This series has \( \frac{2}{n^{3/2}} \), terms decreasing,
\[ \int_1^{\infty} \frac{2}{x^{3/2}} \, dx = -\frac{4}{x^{1/2}} \bigg|_1^{\infty} = 4 \Rightarrow \text{series converges} \]

See 10.3 p598 #22
\[ \sum_{n=1}^{\infty} \frac{5^n}{4^n+3} \]
\[ \lim_{n \to \infty} \frac{5^n}{4^n+3} = \lim_{n \to \infty} \frac{5^n}{4^n} = \infty \]
Series diverges by the \( n \)th term test.

See 10.3 p598 #30
\[ \sum_{n=1}^{\infty} \frac{1}{(\ln n)^n} \]
Since \( \ln 3 > 1 \), this is a geometric series with \( 1/\ln 3 < 1 \), so it converges (and \( \frac{1}{\ln 3} - 1 \)).

b) Each time, a triangle of area \( \frac{A_1}{9^n} \) is added on each side, so the first time \( \frac{3 \cdot A_1}{9} \), then \( 3 \cdot \frac{A_1}{9^2} \) then \( 3 \cdot \frac{A_1}{9^3} \) etc. So the total area is
\[ A_1 \left( 1 + \frac{3}{9} + \frac{3}{9^2} + \frac{3}{9^3} + \cdots \right) \]
\[ = A_1 \left( 1 + \sum_{n=0}^{\infty} \frac{1}{3} (\frac{1}{9})^n \right) \]
\[ \stackrel{\text{Note}}{=} A_1 \left( 1 + \frac{\sqrt{3}}{1-\frac{1}{9}} \right) = A_1 (1 + \frac{\sqrt{3}}{8}) = A_1 (\frac{8 + \sqrt{3}}{5}) = \frac{8}{5} A_1 \]
Sec 10.3 p598 # 34

\[ \sum_{n=1}^{\infty} n \tan^{-1} \frac{1}{n} \]

\[ = \lim_{n \to \infty} n \tan^{-1} \frac{1}{n} \]

\[ = \lim_{n \to \infty} \frac{-\frac{1}{2} \ln(n^2+1)}{1} \]

So series diverges by nth term test.

Sec 10.3 p598 # 38

\[ \sum_{n=1}^{\infty} \frac{n}{n^2+1} \quad \text{terms } \to \infty, \text{ decreases} \]

\[ \therefore \int_{1}^{\infty} \frac{1}{x^2+1} \, dx = \frac{1}{2} \ln(x^2+1) \]

So series diverges.

Sec 10.3 p598 # 60

a) The sum is bounded as follows:

\[ \sum_{k=1}^{10} \frac{1}{k^4} + \int_{10}^{\infty} \frac{1}{x^4} \, dx < \sum_{k=1}^{\infty} \frac{1}{k^4} < \sum_{k=1}^{10} \frac{1}{k^4} + \int_{1}^{10} \frac{1}{x^4} \, dx \]

\[ \approx 1.082036583 < \sum_{k=1}^{\infty} \frac{1}{k^4} < 1.082036583 + 0.000250438 \]

\[ = 1.082287 < \sum_{k=1}^{\infty} \frac{1}{k^4} < 1.082370 \]

b) The difference between the two bounds is about 0.000083, so their average, namely 1.082328468 is off by less than half of this, or 0.0000415.

Sec 10.4 p603 # 4

\[ \sum_{n=1}^{\infty} \frac{n+2}{2n^2-1} \quad \text{Well, } \frac{n+2}{2} > \frac{n}{n} = \frac{1}{n} , \]

and \( \sum \frac{1}{n} \) is the (divergent) harmonic series.

\[ \therefore \sum \frac{n+2}{2n^2-1} \quad \text{diverges by the comparison test.} \]

Sec 10.4 p603 # 6

\[ \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{Well, } \frac{1}{n^3} < \frac{1}{3^n} , \]

and \( \sum \frac{1}{3^n} \) is a convergent geometric series (\( r = \frac{1}{3} < 1 \)).

\[ \therefore \sum \frac{1}{n^3} \quad \text{converges by the comparison test.} \]

Sec 10.4 p603 # 12

\[ \sum_{n=1}^{\infty} \frac{2^n}{3+4^n} \quad \text{(Regular comparison:} \quad \frac{2^n}{3+4^n} < \frac{2^n}{4^n} = \left( \frac{1}{2} \right)^n \quad \text{so this converges by comparison to}\} \]

\[ \text{the geometric series} \}

\[ \text{limit compare with } \sum \frac{2^n}{4^n} = \sum \frac{1}{2^n} \]

\[ \lim_{n \to \infty} \frac{\frac{2^n}{3+4^n}}{\frac{1}{2^n}} = \lim_{n \to \infty} \frac{2^n}{2^n} = \frac{4^n}{3+4^n} = 1 \]

\[ \text{So two sides do the same thing} \]

\[ \text{namely converge since } \sum \frac{1}{2^n} \text{ converges.} \]

Sec 10.4 p603 # 16

\[ \sum \ln(1+\frac{1}{n^2}) \]

\[ \text{Limit comp. to } \sum \frac{1}{n^2} \text{ (Convergent p-series p=2 > 1)} \]

\[ \lim_{n \to \infty} \ln(1+\frac{1}{n^2}) = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n^2}} \]

\[ = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n^2}} = 1 \]

\[ \therefore \sum \ln(1+\frac{1}{n^2}) \]

\[ \text{converges by LCT with } \sum \frac{1}{n^2}. \]
Sec 10.4 p603 #22
\[ \sum_{n=1}^{\infty} \frac{n+1}{n^2 \sqrt{n}} \]

The n-th term is like \( \frac{n}{n^{3/2}} = \frac{1}{n^{1/2}} \)

Limit comparison with \( \sum \frac{1}{n^{1/2}} \) (Convergent p-series \( p = \frac{3}{2} > 1 \))

\[ \lim_{n \to \infty} \frac{n+1}{n^{3/2}} = \lim_{n \to \infty} \frac{n}{n^{3/2}} = 1 \]

So our series converges by L.C.T.

Sec 10.4 p603 #28
\[ \sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3} \]

Since \( \ln n < \sqrt{n} \), compare to \( \sum \frac{1}{n^{3/2}} \) (Convergent p-series \( p = 2 \))

\[ \frac{(\ln n)^2}{n^3} \leq \frac{(\sqrt{n})^2}{n^3} = \frac{1}{n} \]

So \( \sum \frac{(\ln n)^2}{n^3} \) converges by C.T.

Sec 10.4 p603 #34
\[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^{2} + 1} \]

Like \( \frac{\sqrt{n}}{n^{3/2}} = \frac{1}{n^{1/2}} \)

So compare to \( \sum \frac{1}{n^{3/2}} \) (Convergent p-series \( p = 3/2 \))

\[ \frac{\sqrt{n}}{n^{2} + 1} < \frac{\sqrt{n}}{n^{3/2}} = \frac{1}{n^{1/2}} \]

So \( \sum \frac{\sqrt{n}}{n^{2} + 1} \) converges by C.T.

Sec 10.4 p603 #40
\[ \sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n} \]

Compare to \( \sum \frac{2^n + 3^n}{4^n} = \sum \left(2^n + 3^n\right) \)

A sum of two convergent geometric series

\[ r = \frac{2}{4}, \frac{3}{4} \]

\[ \sum \frac{2^n + 3^n}{3^n + 4^n} \] converges by C.T.

Sec 10.4 p603 #52
\[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \]

Converges by C.T.

Sec 10.4 p603 #60
\[ a > 0 \]
\[ \lim_{n \to \infty} n^2 a_n = 0 \]

So for large \( n \) we have \( n^2 a_n < 1 \) or \( a_n < \frac{1}{n^2} \). So compare \( \sum a_n \) with \( \sum \frac{1}{n^2} \), which converges.

So \( \sum a_n \) converges, too.