Math 240 second midterm Exam

No books, paper or any electronic device may be used, other than a hand-written note sheet at most 8.5” × 11” in size. Please turn off your cell phones.

This examination consists of nine (9) long-answer questions. Please show all your work. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please box your answers.

NAME (PRINTED):

INSTRUCTOR:

TA:

Recitation Time:

My signature below certifies that I have complied with the University of Pennsylvania’s code of academic integrity in completing this examination.

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Your signature

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>Total</th>
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1. (10 pts) Convert the differential equation

\[ y'' + 2ty' + y = \cos(t) \]

to a first-order linear system.

Choose a new variable \( z \) and let \( y' = z \). Then the differential equation can be written \( z' + 2tz + y = \cos t \) and we have the system:

\[
\begin{align*}
y' &= z \\
z' &= -y - 2tz + \cos t
\end{align*}
\]

which is a first-order linear system.
2. (13 pts) Determine the general solution to the system
\[ \mathbf{x}' = A\mathbf{x} \]
for the matrix
\[ A = \begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{pmatrix}. \]

Since the matrix has rank 1, two of its eigenvalues are zero. The other is the trace, namely 4. For \( \lambda = 0 \) we have the independent eigenvectors
\[ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \]
and for \( \lambda = 4 \) we have the eigenvector
\[ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]

So the general solution is
\[ \mathbf{x} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \]
3. (10 pts) Solve the initial value problem

\[ x' = Ax, \quad x(0) = x_0 \]

where \( A = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} \) and \( x_0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \).

The characteristic polynomial of \( A \) is \((-1 - \lambda)(-3 - \lambda) - 8 = \lambda^2 + 4\lambda - 5 = (\lambda + 5)(\lambda - 1)\). So the eigenvalues are \( \lambda = -5 \) and \( \lambda = 1 \).

For \( \lambda = -5 \) we need the nullspace of

\[ \begin{bmatrix} 4 & 4 \\ 2 & 2 \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \)

and for \( \lambda = 1 \) we need the nullspace of

\[ \begin{bmatrix} -2 & 4 \\ 2 & -4 \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \)

So the general solution is

\[ x = c_1 e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]

To satisfy the initial conditions we need \( c_1 + 2c_2 = 3 \) and \(-c_1 + c_2 = 0\), so \( c_1 = c_2 = 1 \). The solution of the initial value problem is

\[ x = e^{-5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^t \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]
4. (10 pts) Describe the behavior of the solutions of the system of linear first order differential equations

\[ x' = Ax, \]

as \( t \to \infty \), where \( A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) and \( a, b \) are real numbers such that \( a < 0 \) and \( b > 0 \).

Since \( \text{tr} \ A = 2a < 0 \) and \( \det A = a^2 + b^2 > 0 \), the origin is either a stable node (sink), a degenerate stable node, or a stable spiral point. To see which, we’ll simply calculate the eigenvalues, since that’s not too hard: The characteristic polynomial of \( A \) is \( (a - \lambda)^2 + b^2 = \lambda^2 - 2a\lambda + a^2 + b^2 \) so

\[
\lambda = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2)}}{2} = a \pm bi
\]

so the solutions have terms like \( e^{at} \cos bt \) and \( e^{at} \sin bt \) which will yield stable spirals since \( a < 0 \). They’ll be clockwise since \( b > 0 \).
5. (13 pts) Determine the general solution to the system

\[ \mathbf{x}' = A \mathbf{x} \]

for the matrix

\[
A = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}
\]

The characteristic polynomial of \( A \) is \((\lambda^2 + 1)(\lambda - 2)^2\). For \( \lambda = i \) we need the kernel of

\[
\begin{pmatrix}
-i & -1 & 0 & 0 \\
1 & -i & 0 & 0 \\
0 & 0 & 2-i & 1 \\
0 & 0 & 0 & 2-i
\end{pmatrix}
\]

which is spanned by \[ \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} \].

So for \( \lambda = \pm i \) we need the real and imaginary parts of

\[
e^{it} \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} = (\cos t + i \sin t) \begin{pmatrix} i \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix} + i \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix}.
\]

For \( \lambda = 2 \) we need the kernel of

\[
\begin{pmatrix}
-2 & -1 & 0 & 0 \\
1 & -2 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

which is spanned by \[ \mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \].

We will need a generalized eigenvector for \( \lambda = 2 \), i.e., a \( \mathbf{w} \) with \( (A - 2I)\mathbf{w} = \mathbf{v} \). Such a vector is obviously \[ \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \]. So the general solution of the system is

\[
\mathbf{x} = c_1 \begin{pmatrix} -\sin t \\ \cos t \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{pmatrix} + c_3 e^{2t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c_4 e^{2t} \begin{pmatrix} 0 \\ 0 \\ t \\ 1 \end{pmatrix}.
\]
6. (10 pts) Solve the initial value problem

\[ x' = Ax, \quad x(0) = x_0 \]

where \( A = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \) and \( x_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \).

Characteristic polynomial of \( A \): \((-2 - \lambda)(-4 - \lambda) + 1 = \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2\), so \( \lambda = -3 \) is the only eigenvalue. Now

\[ A + 3I = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad (A - 3I)^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

Therefore \( e^{(A+3I)t} = I + (A + 3I)t \), and so

\[ e^{At} = e^{-3t}((1 + 3t)I + tA) = e^{-3t} \begin{bmatrix} 1 + t & -t \\ t & 1 - t \end{bmatrix}. \]

And the solution of the initial value problem is

\[ e^{At}x = e^{-3t} \begin{bmatrix} 1 + t & -t \\ t & 1 - t \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} te^{-3t} \\ (t - 1)e^{-3t} \end{bmatrix}. \]
7. (12 pts) Compute the matrix exponential $e^{At}$ explicitly for the matrix $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & -2 \\ 1 & 1 & 2 \end{pmatrix}$.

You may use the fact that the characteristic polynomial is $p(\lambda) = -(\lambda + 1)(\lambda - 3)^2$.

[If you find an explicit invertible $3 \times 3$ matrix $C$ and an explicit $3 \times 3$ matrix $X(t)$ whose entries are functions in $t$ such that $e^{At} = C \cdot X(t) \cdot C^{-1}$, you can stop there—you don’t have to carry out the matrix multiplication.]

The eigenvalues are $\lambda = -1$ and $\lambda = 3$. For $\lambda = -1$ we need the kernel of

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 4 & -2 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & -8 \\ 0 & 0 & 0 \end{pmatrix} \text{ which is spanned by } \begin{pmatrix} -7 \\ 4 \\ 1 \end{pmatrix}$$

For $\lambda$ we need the kernel of

$$\begin{pmatrix} -3 & 1 & 3 \\ 2 & 0 & -2 \\ 1 & 1 & -1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3, R_2 \rightarrow R_2 - 2R_1} \begin{pmatrix} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & 4 & 0 \end{pmatrix} \text{ which is spanned by } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Since the double eigenvalue $\lambda = 3$ has only one linearly independent eigenvector $v$, we need to find a $w$ so that $(A - 3I)w = v$, which is pretty easily seen to be $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

We conclude that a matrix $X(t)$ satisfying $X'(t) = AX(t)$ is given by

$$X(t) = \begin{pmatrix} -7e^{-t} & e^{3t} & te^{3t} \\ 4e^{-t} & 0 & e^{3t} \\ e^{-t} & e^{3t} & te^{3t} \end{pmatrix}.$$ 

Now the matrix exponential $E(t) = e^{At}$ satisfies $E'(t) = AE(t)$ and $E(0) = I$. Unfortunately, $X(0) \neq I$, but if we multiply $X(t)$ on the right by $X(0)^{-1}$ the resulting matrix will be $E(t)$.

Now

$$X(0) = \begin{pmatrix} -7 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
and we can calculate $X(0)^{-1}$ by row reduction:

$$
\begin{bmatrix}
-7 & 1 & 0 & 1 & 0 & 0 \\
4 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\xrightarrow{R_1 \leftrightarrow R_3, R_2 \rightarrow R_2 - 4R_1}
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 \\
0 & -4 & 1 & 0 & 1 & -4 \\
0 & 8 & 0 & 1 & 0 & 7 \\
\end{bmatrix}
$$

$$
\xrightarrow{R_3 \rightarrow R_3 + 7R_1}
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 1 & -\frac{1}{4} & 0 & -\frac{1}{4} & 1 \\
0 & 0 & 2 & 1 & 2 & -1 \\
\end{bmatrix}
$$

$$
\xrightarrow{R_3 \rightarrow \frac{1}{2}R_3, R_2 \rightarrow -\frac{1}{4}R_2}
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & \frac{1}{8} & 0 & \frac{7}{8} \\
0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \\
\end{bmatrix}
$$

$$
\xrightarrow{R_3 \rightarrow R_3 + 2R_2, R_2 \rightarrow R_2 + \frac{1}{2}R_3}
\begin{bmatrix}
1 & 0 & 0 & -\frac{1}{8} & 0 & \frac{1}{8} \\
0 & 1 & 0 & \frac{1}{8} & 0 & \frac{7}{8} \\
0 & 0 & 1 & \frac{1}{2} & 1 & -\frac{1}{2} \\
\end{bmatrix}
$$

Thus

$$
e^{tA} = X(t)X(0)^{-1} = 
\begin{bmatrix}
-7e^{-t} & e^{3t} & te^{3t} \\
4e^{-t} & 0 & e^{3t} \\
e^{-t} & e^{3t} & te^{3t} \\
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{8} & 0 & \frac{1}{8} \\
\frac{1}{8} & 0 & \frac{7}{8} \\
\frac{1}{2} & 1 & -\frac{1}{2} \\
\end{bmatrix}
$$

$$
= 
\begin{bmatrix}
\frac{7}{8}e^{-t} + \frac{1}{8}e^{3t} + \frac{1}{2}te^{3t} & te^{3t} & -\frac{7}{8}e^{-t} + \frac{7}{8}e^{3t} - \frac{1}{2}te^{3t} \\
-\frac{1}{2}e^{-t} + \frac{1}{2}e^{3t} & e^{3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{3t} \\
-\frac{1}{8}e^{-t} + \frac{1}{8}e^{3t} + \frac{1}{2}te^{3t} & te^{3t} & \frac{1}{8}e^{-t} + \frac{7}{8}e^{3t} - \frac{1}{2}te^{3t} \\
\end{bmatrix}.
$$
8. (12 pts) For each of the three systems \( \mathbf{x}' = A\mathbf{x} \) below, characterize/classify the equilibrium point as stable or unstable node, stable or unstable spiral, center, saddle point, proper node, degenerate node, etc. For any of these three systems, you may opt to sketch the phase portrait instead of characterizing its equilibrium point.

(a) \( A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \)

Since \( \det A = -1 < 0 \), the origin is a saddle point.

(b) \( A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \)

Characteristic polynomial is \((2 - \lambda)^2 + 1 = \lambda^2 - 4\lambda + 5\) so eigenvalues are \( \lambda = \frac{1}{2}(4 \pm \sqrt{16 - 20}) = 2 \pm i \). So this is an unstable spiral point.

(c) \( A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \)

Eigenvalues are \(-1\) and \(-3\) so origin is a stable node (sink).
9. (10 pts) For the system

\[ x' = x(2 - y), \quad y' = y(x + 1) \]

determine all equilibrium points and characterize/classify each equilibrium point as in problem 8.

For \( x' = 0 \) we need either \( x = 0 \) or \( y = 2 \), and for \( y' = 0 \) we need either \( y = 0 \) or \( x = -1 \). So the equilibrium (critical) points, where both are zero, are \((0,0)\) and \((-1,2)\). The Jacobian of the system is

\[ J = \begin{bmatrix} 2 - y & -x \\ y & x + 1 \end{bmatrix} \]

so

\[ J(0,0) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad J(-1,2) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}. \]

The eigenvalues of \( J(0,0) \) are clearly 2 and 1 so \((0,0)\) is an unstable node (a source). The determinant of \( J(-1,2) \) is negative so \((-1,2)\) is a saddle point (unstable).