No books, paper or any electronic device may be used, other than a hand-written note sheet at most 8.5" × 11" in size. Please turn off your cell phones.

This examination consists of nine (9) long-answer questions. Please show all your work. Merely displaying some formulas is not sufficient ground for receiving partial credits. Please box your answers.

NAME (PRINTED):

INSTRUCTOR:

TA:

RECITATION TIME:

My signature below certifies that I have complied with the University of Pennsylvania’s code of academic integrity in completing this examination.

Your signature

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1. (10 pts) Convert the differential equation

\[ y'' + 2ty' + y = \cos(t) \]

to a first-order linear system.
2. (13 pts) Determine the general solution to the system

\[ \mathbf{x}' = A\mathbf{x} \]

for the matrix \( A = \begin{pmatrix} 2 & -1 & 3 \\ 2 & -1 & 3 \\ 2 & -1 & 3 \end{pmatrix} \).
3. (10 pts) Solve the initial value problem

\[ x' = Ax, \ x(0) = x_0 \]

where \( A = \begin{pmatrix} -1 & 4 \\ 2 & -3 \end{pmatrix} \) and \( x_0 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \).
4. (10 pts) Describe the behavior of the solutions of the system of linear first order differential equations 

\[ \mathbf{x}' = A\mathbf{x}, \]

as \( t \to \infty \), where \( A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \) and \( a, b \) are real numbers such that \( a < 0 \) and \( b > 0 \).
5. (13 pts) Determine the general solution to the system

\[ x' = Ax \]

for the matrix

\[
A = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]
6. (10 pts) Solve the initial value problem

\[ x' = Ax, \quad x(0) = x_0 \]

where \( A = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \) and \( x_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \).
7. (12 pts) Compute the matrix exponential $e^{At}$ explicitly for the matrix $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & -2 \\ 1 & 1 & 2 \end{pmatrix}$.

You may use the fact that the characteristic polynomial is $p(\lambda) = -(\lambda + 1)(\lambda - 3)^2$.

[If you find an explicit invertible $3 \times 3$ matrix $C$ and an explicit $3 \times 3$ matrix $X(t)$ whose entries are functions in $t$ such that $e^{At} = C \cdot X(t) \cdot C^{-1}$, you can stop there—you don’t have to carry out the matrix multiplication.]
8. (12 pts) For each of the three systems $\mathbf{x}' = A\mathbf{x}$ below, characterize/classify the equilibrium point as stable or unstable node, stable or unstable spiral, center, saddle point, proper node, degenerate node, etc. For any of these three systems, you may opt to sketch the phase portrait instead of characterizing its equilibrium point.

(a) $A = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

(c) $A = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$
9. (10 pts) For the system

\[ x' = x(2 - y), \quad y' = y(x + 1) \]

determine all equilibrium points and characterize/classify each equilibrium point as in problem 8.