1. Find the general solution of the system

\[ \begin{align*}
    x'(t) &= x + y \\
    y'(t) &= 4x - 2y
\end{align*} \]

Then find the solution satisfying the initial conditions \( x(0) = 1 \), \( y(0) = 6 \). Sketch the graph of this solution.

2. (a) Find the general solution of the system

\[ \begin{pmatrix}
    x' \\
    y'
\end{pmatrix} = \begin{bmatrix}
    1 & 0 & -2 \\
    0 & 1 & 0 \\
    1 & -1 & -1
\end{bmatrix} \begin{pmatrix}
    x \\
    y
\end{pmatrix}. \]

(b) For which initial vectors \( \mathbf{a} \) will the initial-value problem consisting of the system of equations in part (a) together with the initial condition \( \mathbf{x}(0) = \mathbf{a} \) have a solution that is periodic in time?

3. Suppose \( \lambda = a + bi \) with \( b \neq 0 \) and that \( \mathbf{x} = e^{\lambda t} \mathbf{v} \) (where \( \lambda \) is the aforementioned complex number and \( \mathbf{v} \) is a constant (complex) vector) is a solution of \( \mathbf{x}' = A \mathbf{x} \) for some real matrix \( A \).

   (a) Explain why \( \overline{\mathbf{x}} = e^{\bar{\lambda} t} \overline{\mathbf{v}} \) is also a solution of \( \mathbf{x}' = A \mathbf{x} \) (where the bars stand for complex conjugates).

   (b) Explain why \( \text{Re}(\mathbf{x}) \) and \( \text{Im}(\mathbf{x}) \) are also solutions of \( \mathbf{x}' = A \mathbf{x} \), where \( \text{Re}(\mathbf{x}) \) is the real part of \( \mathbf{x} \) and \( \text{Im}(\mathbf{x}) \) is the imaginary part of \( \mathbf{x} \).

   (c) Explain why \( \text{Re}(\mathbf{x}) \) and \( \text{Im}(\mathbf{x}) \) are linearly independent (Read about the chapter 7 version of the Wronskian in the textbook, and use it).

   (d) Solve

\[ \begin{align*}
    x' &= 4x - 2y \\
    y' &= 5x - 2y
\end{align*} \]

   together with the initial conditions \( x(0) = 1 \) and \( y(0) = 2 \).

4. (a) Suppose the radioactive element \( A \) decays into the element \( B \) with a half-life of 6 hours. Explain why this implies that

\[ \frac{dA}{dt} = -\frac{\ln 2}{6} A \]
where $A$ is the amount of substance $A$ in grams at time $t$ hours.

(b) Also explain why this means that

$$\frac{dB}{dt} = -\frac{\ln 2}{6} A$$

where $B(t)$ is the amount of substance $B$ in grams at time $t$ hours.

(c) Now also suppose that the element $B$ decays into the element $C$ with a half-life of 9 hours. Explain why the amounts $A(t)$, $B(t)$ and $C(t)$ respectively satisfy the system of differential equations

\[
\begin{align*}
A' &= -\frac{\ln 2}{6} A \\
B' &= \frac{\ln 2}{6} A - \frac{\ln 2}{9} B \\
C' &= \frac{\ln 2}{9} B
\end{align*}
\]

(d) Solve this system together with the initial conditions $A(0) = 100$, $B(0) = 0$ and $C(0) = 0$. Plot the graphs of $A(t)$, $B(t)$ and $C(t)$ on the same graph, with $t$ on the horizontal axis and $A$, $B$, and $C$ on the vertical. Then answer the following questions:

What is $\lim_{t \to \infty} A(t)$? How about $\lim_{t \to \infty} C(t)$? And $\lim_{t \to \infty} B(t)$?

At (approximately) what time $t$ does the maximum of $B$ occur? What is the maximum?

5. In class on Thursday, we started to think about what happens when we are trying to find the general solution of $\mathbf{x}' = A\mathbf{x}$ and $A$ has an eigenvalue $\lambda_0$ that is a double root of $\det(A - \lambda I)$ but which has only one linearly independent eigenvector $\mathbf{v}_0$ (i.e., the space of solutions of $(A - \lambda_0 I)\mathbf{v} = \mathbf{0}$ is only one-dimensional). So we know that $\mathbf{x} = e^{\lambda_0 t}\mathbf{v}_0$ is a solution of the system of differential equations, but we need to find a second, independent solution and don’t have anything at hand to make it out of.

However, in this case, it is always true that there will be a vector $\mathbf{w}_0$, linearly independent from $\mathbf{v}_0$, that satisfies $(A - \lambda_0 I)\mathbf{w}_0 = \mathbf{v}_0$. It is standard to call $\mathbf{w}_0$ a “generalized eigenvector” of $A$ corresponding to the eigenvalue $\lambda_0$, since $(A - \lambda_0 I)^2\mathbf{w}_0 = \mathbf{0}$.

(a) Prove this last statement. I.e., show that if $(A - \lambda_0 I)\mathbf{w}_0 = \mathbf{v}_0$, then $(A - \lambda_0 I)^2\mathbf{w}_0 = \mathbf{0}$.

(b) Show that, in this case $\mathbf{x} = e^{\lambda_0 t}(t\mathbf{v}_0 + \mathbf{w}_0)$ is a linearly independent solution of $\mathbf{x}' = A\mathbf{x}$.

(c) Use part (b) to find the general solution of $\mathbf{x}' = A\mathbf{x}$ for $A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix}$.