Make sure you can answer all of the True-False review questions at the end of sections 7.1, 7.2, 7.3 and 7.4 of the textbook. Also, make sure you can do the “core problems” (section 7.1: 5, 9, 11, 15, 18, 20; section 7.2: 5, 7, 12, 13; section 7.3: 2, 5, 7; and section 7.4: 6, 17, 19, 20, 23, 26).

Then write up solutions to the following to hand in on Tuesday April 7:

1. Find the general solution of the system

\[ \begin{align*}
x'(t) &= x + y \\
y'(t) &= 4x - 2y
\end{align*} \]

Then find the solution satisfying the initial conditions \( x(0) = 1, \ y(0) = 6 \). Sketch the graph of this solution.

We’ll solve this one three different ways: by “algebraic” manipulation, by the eigenvalue/eigenvector method, and using the matrix exponential function.

First, we can be clever about the algebra as follows: use \( D \) to represent the derivative with respect to \( t \), and rewrite the system as:

\[
\begin{align*}
(D - 1)x - y &= 0 \\
-4x + (D + 2)y &= 0
\end{align*}
\]

“Multiply” the first equation by \( D + 2 \) (that is, take the derivative and add two times what is there) to replace the first equation with

\[
(D + 2)(D - 1)x - (D + 2)y = 0.
\]

Now, \((D + 2)(D - 1)x = (D^2 + D - 2)x\), where \( D^2 \) means the second derivative with respect to \( t \). Add the second equation to this to get:

\[
(D^2 + D - 2)x - 4x = x'' + x' - 6x = 0.
\]

Since the solutions of the polynomial equation \( r^2 + r - 6 = 0 \) are \( r = 2 \) and \( r = -3 \), we have that \( x(t) = c_1 e^{2t} + c_2 e^{-3t} \). Then the first equation tells us that

\[
y = x' - x = (2c_1 e^{2t} - 3c_2 e^{-3t}) - (c_1 e^{2t} + c_2 e^{-3t}) = c_1 e^{2t} - 4c_2 e^{3t}.
\]

So the general solution of the system is

\[
\begin{align*}
x &= c_1 e^{2t} + c_2 e^{-3t} \\
y &= c_1 e^{2t} - 4c_2 e^{3t}
\end{align*}
\]

To match the initial conditions, we note that \( x(0) = c_1 + c_2 \) and \( y(0) = c_1 - 4c_2 \). So we need \( c_1 + c_2 = 1 \) and \( c_1 - 4c_2 = 6 \). Solve this algebraic system and get \( c_1 = 2 \) and \( c_2 = -1 \). So the unique solution of the initial value problem is

\[
\begin{align*}
x &= 2e^{2t} - e^{-3t} \\
y &= 2e^{2t} + 4e^{-3t}
\end{align*}
\]
Next, we try the eigenvalue/eigenvector method. Rewrite the system as \( \mathbf{x}' = A \mathbf{x} \) where

\[
A = \begin{bmatrix}
1 & 1 \\
4 & -2
\end{bmatrix}.
\]

Then

\[
\det(A - \lambda I) = \det \begin{bmatrix}
1 - \lambda & 1 \\
4 & -2 - \lambda
\end{bmatrix} = (1 - \lambda)(-2 - \lambda) - 4 = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2)
\]

So the eigenvalues are \( \lambda = 2 \) and \( \lambda = -3 \). For \( \lambda = 2 \) we need to find the nullspace of

\[
A - 2I = \begin{bmatrix}
-1 & 1 \\
4 & -4
\end{bmatrix}
\]

which is spanned by the vector \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). For \( \lambda = -3 \) we need the nullspace of

\[
A + 3I = \begin{bmatrix}
4 & 1 \\
4 & 1
\end{bmatrix}
\]

which is spanned by the vector \( \begin{bmatrix} 1 \\ -4 \end{bmatrix} \). So we can write the general solution of \( \mathbf{x}' = A \mathbf{x} \) as

\[
\mathbf{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -4 \end{bmatrix}.
\]

This is the same as the general solution we got by the previous method, and the rest of the solution proceeds the same way as before.

Finally, we try the matrix exponential method, for which we need to calculate \( e^{tA} \), where \( A \) is the matrix given above. From the eigenvalue/eigenvector version of the solution, we already know that \( A = PDP^{-1} \), where

\[
D = \begin{bmatrix}
2 & 0 \\
0 & -3
\end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix}
1 & 1 \\
1 & -4
\end{bmatrix}.
\]

Since \( \det P = -5 \), we can easily calculate that

\[
P^{-1} = \begin{bmatrix}
\frac{4}{5} & \frac{1}{5} \\
\frac{1}{5} & -\frac{1}{5}
\end{bmatrix}.
\]

Therefore

\[
e^{tA} = Pe^{tD}P^{-1} = \begin{bmatrix}
1 & 1 \\
1 & -4
\end{bmatrix} \begin{bmatrix}
e^{2t} & 0 \\
0 & e^{-3t}
\end{bmatrix} \begin{bmatrix}
\frac{4}{5} & \frac{1}{5} \\
\frac{1}{5} & -\frac{1}{5}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
e^{2t} & e^{-3t} \\
e^{2t} & -4e^{-3t}
\end{bmatrix} \begin{bmatrix}
\frac{4}{5} & \frac{1}{5} \\
\frac{1}{5} & -\frac{1}{5}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{4}{5} e^{2t} + \frac{1}{5} e^{-3t} \\
\frac{4}{5} e^{2t} - \frac{1}{5} e^{-3t}
\end{bmatrix} \begin{bmatrix}
\frac{4}{5} e^{2t} - \frac{1}{5} e^{-3t} \\
\frac{4}{5} e^{2t} + \frac{4}{5} e^{-3t}
\end{bmatrix}.
\]
From this, we get the general solution by multiplying by an arbitrary constant vector \( \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \):

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = \begin{bmatrix}
  \frac{4}{5}e^{2t} + \frac{2}{5}e^{-3t} \\
  \frac{2}{5}e^{2t} - \frac{3}{5}e^{-3t}
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2
\end{bmatrix} = \begin{bmatrix}
  \left( \frac{4}{5}a_1 + \frac{2}{5}a_2 \right)e^{2t} + \left( \frac{2}{5}a_1 - \frac{3}{5}a_2 \right)e^{-3t} \\
  \left( \frac{2}{5}a_1 + \frac{3}{5}a_2 \right)e^{2t} + \left( \frac{4}{5}a_1 - \frac{2}{5}a_2 \right)e^{-3t}
\end{bmatrix}
\]

This agrees with the general solution we found above if we set \( c_1 = \frac{4}{5}a_1 + \frac{2}{5}a_2 \) and \( c_2 = \frac{2}{5}a_1 - \frac{3}{5}a_2 \). We get the solution to the initial-value problem by setting \( a_1 = x(0) = 1 \) and \( a_2 = y(0) = 6 \):

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} = \begin{bmatrix}
  2e^{2t} - e^{-3t} \\
  2e^{2t} + 4e^{-3t}
\end{bmatrix}.
\]

The graph of the solution should be tangent to the line \( y = x \) as \( t \to +\infty \) since the \( e^{-3t} \) terms will be tiny, and to the line \( y = -4x \) as \( t \to -\infty \) since then the \( e^{2t} \) terms will be negligible. Here’s the graph (with the two asymptotes in red):

![Figure 1: Solution of \( x' = x + y, y' = 4x - 2y, x(0) = 1, y(0) = 6 \), with asymptotes](image)

2. (a) Find the general solution of the system

\[
x' = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{bmatrix} x.
\]

(b) For which initial vectors \( \mathbf{a} \) will the initial-value problem consisting of the system of equations in part (a) together with the initial condition \( \mathbf{x}(0) = \mathbf{a} \) have a solution that is periodic in time?

(a) We’ll do this one three different ways, too. For the first way, we’ll write the system as:

\[
x' = x - 2z
\]

\[
y' = y
\]

\[
z' = x - y - z
\]
The second equation involves only $y$, and its general solution is $y = c_1 e^t$. So now we can rewrite the first and third equations as

\[
(D - 1)x + 2z = 0 \\
-x + (D + 1)z = -c_1 e^t
\]

If we apply $D - 1$ to the second equation (and notice that $(D - 1)e^t = 0$) and then add the two equations together we eliminate $x$ and get that $z$ satisfies:

\[
2z + (D - 1)(D + 1)z = z'' + z = 0.
\]

So $z = c_2 \cos t + c_3 \sin t$. Then the second equation above gives us that

\[
x = c_1 e^t + (D + 1)z = c_1 e^t + (-c_2 \sin t + c_3 \cos t) + (c_2 \cos t + c_3 \sin t) = c_1 e^t + (c_2 + c_3) \cos t - (c_2 - c_3) \sin t.
\]

Altogether, the general solution is

\[
x = c_1 e^t + (c_2 + c_3) \cos t - (c_2 - c_3) \sin t \\
y = c_1 e^t \\
z = c_2 \cos t + c_3 \sin t
\]

Next we do the eigenvalue/eigenvector method. Let $A$ be the matrix in the problem and calculate

\[
\det(A - \lambda I) = \det \begin{bmatrix}
1 - \lambda & 0 & -2 \\
0 & 1 - \lambda & 0 \\
1 & -1 & -1 - \lambda
\end{bmatrix}
= (1 - \lambda)(1 - \lambda)(-1 - \lambda) + 2(1 - \lambda)
= (1 - \lambda)((1 - \lambda)(-1 - \lambda) + 2)
= (1 - \lambda)(\lambda^2 + 1)
\]

so the eigenvalues of $A$ are $1$, $i$ and $-i$. For $\lambda = 1$ we need to find the nullspace of

\[
A - I = \begin{bmatrix}
0 & 0 & -2 \\
0 & 0 & 0 \\
1 & -1 & -2
\end{bmatrix}
\]

which is spanned by $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

For $\lambda = i$, we need the nullspace of

\[
A - iI = \begin{bmatrix}
1 - i & 0 & -2 \\
0 & 1 - i & 0 \\
1 & -1 & -1 - i
\end{bmatrix}
\]

which is spanned by $\begin{bmatrix} 1 + i \\ 0 \\ 1 \end{bmatrix}$

The solution that corresponds to $\lambda = 1$ is $e^{it} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and the solutions that correspond to $\lambda = \pm i$ are the real and imaginary parts of

\[
e^{it} \begin{bmatrix} 1 + i \\ 0 \\ 1 \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1 + i \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} (\cos t - \sin t) + i(\cos t + \sin t) \\ 0 \\ \cos t + i \sin t \end{bmatrix}
\]
so the two solutions are

\[
\begin{bmatrix}
\cos t - \sin t \\
0 \\
\cos t
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
\cos t + \sin t \\
0 \\
\sin t
\end{bmatrix}.
\]

We write the general solution as

\[
x = c_1 \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} \cos t - \sin t \\ 0 \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t + \sin t \\ 0 \\ \sin t \end{bmatrix}.
\]

which agrees with what we got before.

Finally, we do the matrix exponential method. We already know that the eigenvalues of \( A \) are 1, \( i \) and \(-i\) and we have the eigenvectors for 1 and \( i \), and for \( \lambda = -i \) we have that the nullspace of

\[
A + iI = \begin{bmatrix} 1 + i & 0 & -2 \\ 0 & 1 + i & 0 \\ 1 & -1 & -1 + i \end{bmatrix}
\]
is spanned by \begin{bmatrix} 1 - i \\ 0 \\ 1 \end{bmatrix}.

Therefore, \( A = PDP^{-1} \) where

\[
D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}
\quad \text{and} \quad
P = \begin{bmatrix} 1 & 1 + i & 1-i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.
\]

We’re going to need \( P^{-1} \), so we row-reduce:

\[
\begin{bmatrix}
1 & 1 & 1-i \\
1 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}
\xrightarrow{R_1 \leftrightarrow R_2, \: R_2 \leftrightarrow R_3}
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 0
\end{bmatrix}
\xrightarrow{R_3 \rightarrow R_3 - (1+i)R_2}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & -2i
\end{bmatrix}
\xrightarrow{R_3 \rightarrow R_3 + (1+i)R_2}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\xrightarrow{R_3 \rightarrow \frac{1}{2} i R_3}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{2} + \frac{1}{2}i
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - R_3}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{2} + \frac{1}{2}i
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - R_3}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{2} + \frac{1}{2}i
\end{bmatrix}
\]

Therefore

\[
P^{-1} = \begin{bmatrix}
0 & 1 & 0 \\
-\frac{1}{2}i & \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \\
\frac{1}{2}i & -\frac{1}{2}i & \frac{1}{2} - \frac{1}{2}i
\end{bmatrix}.
\]
and so

\[ e^{tA} = P e^{tD} P^{-1} = \begin{bmatrix} 1 & 1 + i & 1 - i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{it} & 0 \\ 0 & 0 & e^{-it} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2}i & \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \\ \frac{1}{2}i & -\frac{1}{2}i & \frac{1}{2} - \frac{1}{2}i \end{bmatrix} \]

\[ = \begin{bmatrix} e^t & (1 + i)e^{it} & (1 - i)e^{-it} \\ e^t & 0 & 0 \\ 0 & e^{it} & e^{-it} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -\frac{1}{2}i & \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \\ \frac{1}{2}i & -\frac{1}{2}i & \frac{1}{2} - \frac{1}{2}i \end{bmatrix} \]

\[ = \begin{bmatrix} (\frac{1}{2} - \frac{1}{2}i)e^{it} + (\frac{1}{2} + \frac{1}{2}i)e^{-it} & e^t + (-\frac{1}{2} + \frac{1}{2}i)e^{it} + (-\frac{1}{2} - \frac{1}{2}i)e^{-it} & ie^{it} - ie^{-it} \\ 0 & e^t & 0 \\ -\frac{1}{2}ie^{it} + \frac{1}{2}ie^{-it} & \frac{1}{2}ie^{it} - \frac{1}{2}ie^{-it} & (\frac{1}{2} + \frac{1}{2}i)e^{it} + (\frac{1}{2} - \frac{1}{2}i)e^{-it} \end{bmatrix}. \]

It is remarkable (but not surprising if you think about it) that this last matrix is real. We can calculate its real form by recalling Euler’s formula and its consequences:

\[ \cos t = \frac{1}{2} e^{it} + \frac{1}{2} e^{-it} \quad \text{and} \quad \sin t = -\frac{1}{2} ie^{it} + \frac{1}{2} ie^{-it}. \]

Using these we see that

\[ e^{tA} = \begin{bmatrix} \cos t + \sin t & e^t - \cos t - \sin t & -2 \sin t \\ 0 & e^t & 0 \\ \sin t & -\sin t & \cos t - \sin t \end{bmatrix} \]

From this, we get that the solution of the differential system \( \dot{x}' = Ax \) with initial conditions \( x(0) = a \) is

\[ x = e^{tA}a = \begin{bmatrix} \cos t + \sin t & e^t - \cos t - \sin t & -2 \sin t \\ 0 & e^t & 0 \\ \sin t & -\sin t & \cos t - \sin t \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \]

\[ = \begin{bmatrix} a_2 e^t + (a_1 - a_2) \cos t + (a_1 - a_2 - 2a_3) \sin t \\ a_3 \cos t + (a_1 - a_2 - a_3) \sin t \\ a_2 e^t \end{bmatrix}. \]

This agrees with what we got before, where \( c_1 = a_2, c_2 = a_3 \) and \( c_3 = a_1 - a_2 - a_3 \).

(b) The solution will be periodic (with period 2\( \pi \)) if there are no \( e^t \) terms. From the matrix exponential version of the solution, we see immediately that this happens when \( a_2 = 0 \), so the initial vector \( a \) should be of the form:

\[ a = \begin{bmatrix} a_1 \\ 0 \\ a_3 \end{bmatrix} \]

for arbitrary \( a_1 \) and \( a_3 \).

3. Suppose \( \lambda = a + bi \) with \( b \not= 0 \) and that \( x = e^{\lambda t}\mathbf{v} \) (where \( \lambda \) is the aforementioned complex number and \( \mathbf{v} \) is a constant (complex) vector) is a solution of \( \dot{x}' = Ax \) for some real matrix \( A \).

(a) Explain why \( \overline{x} = e^{\overline{\lambda} t}\overline{\mathbf{v}} \) is also a solution of \( \dot{x}' = Ax \) (where the bars stand for complex conjugates).
(b) Explain why \( \text{Re}(\mathbf{x}) \) and \( \text{Im}(\mathbf{x}) \) are also solutions of \( \mathbf{x}' = A\mathbf{x} \), where \( \text{Re}(\mathbf{x}) \) is the real part of \( \mathbf{x} \) and \( \text{Im}(\mathbf{x}) \) is the imaginary part of \( \mathbf{x} \).

(c) Explain why \( \text{Re}(\mathbf{x}) \) and \( \text{Im}(\mathbf{x}) \) are linearly independent (Read about the chapter 7 version of the Wronskian in the textbook, and use it).

(d) Solve
\[
\begin{align*}
\mathbf{x}' &= 4\mathbf{x} - 2\mathbf{y} \\
\mathbf{y}' &= 5\mathbf{x} - 2\mathbf{y}
\end{align*}
\]
together with the initial conditions \( x(0) = 1 \) and \( y(0) = 2 \).

(a) Two ways to do this – a more sophisticated approach is first to observe that \( e^{\lambda t} = e^{\overline{\lambda}t} \) if \( t \) is real, since
\[
e^{\lambda t} = e^{(a-ib)t} = e^{at}(\cos bt - i \sin bt) = e^{at}(\cos t + i \sin t) = e^{(a+ib)t} = e^{\overline{\lambda}t}.
\]
Since the matrix \( A \) is real, if \( \mathbf{x}' = A\mathbf{x} \), then, taking the complex conjugate of both sides we get
\[
\overline{\mathbf{x}}' = A\overline{\mathbf{x}}, \quad \text{or} \quad \mathbf{x}' = A\overline{\mathbf{x}},
\]
so \( \mathbf{x} \) is also a solution.

A more prosaic approach is to start by letting \( \mathbf{u} = \text{Re}(\mathbf{v}) \) and \( \mathbf{w} = \text{Im}(\mathbf{v}) \), so \( \mathbf{v} = \mathbf{u} + i\mathbf{w} \), and then note that
\[
\mathbf{x} = e^{\lambda t} \mathbf{v} = e^{at}(\cos bt + i \sin bt)(\mathbf{u} + i\mathbf{w}) = e^{at}\left( ((\cos bt)\mathbf{u} - (\sin bt)\mathbf{w}) + i(\sin bt)\mathbf{u} + (\cos bt)\mathbf{w} \right).
\]
We know that \( \mathbf{v} \) is an eigenvector of \( A \) with eigenvalue \( \lambda \). This means that \( A\mathbf{v} = \lambda \mathbf{v} \), or in other words \( A(\mathbf{u} + i\mathbf{w}) = (a + ib)(\mathbf{u} + i\mathbf{w}) \) which is to say
\[
A\mathbf{u} + iA\mathbf{w} = (a\mathbf{u} - b\mathbf{w}) + i(a\mathbf{w} + b\mathbf{u}),
\]
which implies that \( A\mathbf{u} = a\mathbf{u} - b\mathbf{w} \) and \( A\mathbf{w} = a\mathbf{w} + b\mathbf{u} \). Then we can check that
\[
(\overline{\mathbf{x}})' = \left( e^{\overline{\lambda}t} \mathbf{v} \right)' = \frac{d}{dt} \left( e^{at}(\cos bt - i \sin bt)(\mathbf{u} - i\mathbf{w}) \right)
\]
\[
= e^{at}\left( (a \cos bt - b \sin bt) - i( a \sin bt + b \cos bt) \right)(\mathbf{u} - i\mathbf{w})
\]
\[
= e^{at}\left( [(a \cos bt - b \sin bt)\mathbf{u} - (a \sin bt + b \cos bt)\mathbf{w}] - i[(a \sin bt + b \cos bt)\mathbf{u} + (a \cos bt - b \sin bt)\mathbf{w}] \right)
\]
which is the same as
\[
A\overline{\mathbf{x}} = e^{at}(\cos bt - i \sin bt)(A\mathbf{u} - iA\mathbf{w})
\]
\[
= e^{at}(\cos bt - i \sin bt)\left( (a\mathbf{u} - b\mathbf{w}) - i(b\mathbf{u} + a\mathbf{w}) \right)
\]
\[
= e^{at}\left( [(a \cos bt - b \sin bt)\mathbf{u} - (a \sin bt + b \cos bt)\mathbf{w}] - i[(a \sin bt + b \cos bt)\mathbf{u} + (a \cos bt - b \sin bt)\mathbf{w}] \right).
\]

(b) Any linear combination of solutions to the linear equation \( \mathbf{x}' = A\mathbf{x} \) will also be a solution. Therefore, since \( \mathbf{x} \) and \( \overline{\mathbf{x}} \) are both solutions, \( \text{Re}(\mathbf{x}) = \frac{1}{2}(\mathbf{x} + \overline{\mathbf{x}}) \) and \( \text{Im}(\mathbf{x}) = \frac{1}{2}i(\overline{\mathbf{x}} - \mathbf{x}) \) are also solutions.
(c) There are two steps to showing that Re($\mathbf{x}$) and Im($\mathbf{x}$) are linearly independent. The first is to derive the consequence of the fact that $\mathbf{v} = \mathbf{u} + i\mathbf{w}$ and $\mathbf{v} = \mathbf{u} - i\mathbf{w}$ are linearly independent, because they are eigenvectors corresponding to two different eigenvalues. This means that the determinant of the matrix with $\mathbf{v}$ as its first column and $\mathbf{v}$ as its second column is not zero. Let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$ 

Then

$$0 \neq \det \begin{bmatrix} u_1 + iw_1 & u_1 - iw_1 \\ u_2 + iw_2 & u_2 - iw_2 \end{bmatrix} = (u_1 + iw_1)(u_2 - iw_2) - (u_1 - iw_1)(u_2 + iw_2) = 2i(u_2w_1 - u_1w_2).$$

Therefore the quantity $u_1w_2 - u_2w_1$ is a non-zero constant.

Now we calculate the Wronskian of Re($\mathbf{x}$) and Im($\mathbf{x}$):

$$W(\text{Re}(\mathbf{x}), \text{Im}(\mathbf{x})) = \det \begin{bmatrix} e^{at}(u_1 \cos bt - w_1 \sin bt) & e^{at}(u_1 \sin bt + w_1 \cos bt) \\ e^{at}(u_2 \cos bt - w_2 \sin bt) & e^{at}(u_2 \sin bt + w_2 \cos bt) \end{bmatrix}$$

$$= e^{2at} \begin{bmatrix} (u_1 \cos bt - w_1 \sin bt)(u_2 \sin bt + w_2 \cos bt) - (u_1 \sin bt + w_1 \cos bt)(u_2 \cos bt - w_2 \sin bt) \end{bmatrix}$$

which we know is nonzero by the above paragraph. Since the Wronskian of Re($\mathbf{x}$) and Im($\mathbf{x}$) is thus never zero, we know that Re($\mathbf{x}$) and Im($\mathbf{x}$) are linearly independent for all $t$.

(d) Let

$$A = \begin{bmatrix} 4 & -2 \\ 5 & -2 \end{bmatrix}$$

be the matrix in the problem. We need the eigenvalues and eigenvectors of $A$. Since $\det(A - \lambda I) = (4 - \lambda)(-2 - \lambda) + 10 = \lambda^2 - 2\lambda + 2$, we have

$$\lambda = \frac{2 \pm \sqrt{4 - 8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i.$$

To find an eigenvalue for $\lambda = 1 + i$ we need the nullspace of

$$A - (1 + i)I = \begin{bmatrix} 3 - i & -2 \\ 5 & -3 - i \end{bmatrix}$$

which is spanned by $\begin{bmatrix} 2 \\ 3 - i \end{bmatrix}$.

From part (b), we know that a basis for the solution space consists of the real and imaginary parts of

$$e^{(1+i)t} \begin{bmatrix} 2 \\ 3 - i \end{bmatrix} = e^t(\cos t + i \sin t) \begin{bmatrix} 2 \\ 3 - i \end{bmatrix} = \begin{bmatrix} 2e^t \cos t + 2ie^t \sin t \\ (3e^t \cos t + e^t \sin t) + i(3e^t \sin t - e^t \cos t) \end{bmatrix}.$$

Therefore the general solution is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 2e^t \cos t \\ 3e^t \cos t + e^t \sin t \end{bmatrix} + c_2 \begin{bmatrix} 2e^t \sin t \\ 3e^t \sin t - e^t \cos t \end{bmatrix}.$$ 

Since

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 2c_1 \\ 3c_1 - c_2 \end{bmatrix}$$
and we want \( x(0) = 1 \) and \( y(0) = 2 \), we see that \( c_1 = \frac{1}{2} \) and \( c_2 = -\frac{1}{2} \). So the solution of the initial-value problem is

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e^t \cos t - e^t \sin t \\ 2e^t \cos t - e^t \sin t \end{bmatrix}.
\]

4. (a) Suppose the radioactive element \( A \) decays into the element \( B \) with a half-life of 6 hours. Explain why this implies that

\[
\frac{dA}{dt} = -\frac{\ln 2}{6} A
\]

where \( A \) is the amount of substance \( A \) in grams at time \( t \) hours.

(b) Also explain why this means that

\[
\frac{dB}{dt} = +\frac{\ln 2}{6} A
\]

where \( B(t) \) is the amount of substance \( B \) in grams at time \( t \) hours.

(c) Now also suppose that the element \( B \) decays into the element \( C \) with a half-life of 9 hours. Explain why the amounts \( A(p), B(p), \) and \( C(p) \) respectively satisfy the system of differential equations

\[
\begin{align*}
A' &= -\frac{\ln 2}{6} A \\
B' &= \frac{\ln 2}{6} A - \frac{\ln 2}{9} B \\
C' &= \frac{\ln 2}{9} B
\end{align*}
\]

(d) Solve this system together with the initial conditions \( A(0) = 100, B(0) = 0 \) and \( C(0) = 0 \). Plot the graphs of \( A(t), B(t), \) and \( C(t) \) on the same graph, with \( t \) on the horizontal axis and \( A, B, \) and \( C \) on the vertical. Then answer the following questions:

What is \( \lim_{t \to \infty} A(t) \)? How about \( \lim_{t \to \infty} C(t) \)? And \( \lim_{t \to \infty} B(t) \)?

At (approximately) what time \( t \) does the maximum of \( B \) occur? What is the maximum?

(a) The radioactive decay of element \( A \) is an exponential decay process, so we know that \( A' = -kA \), and so \( A = 100e^{-kt} \). Since half of the initial amount of \( A \) is left at time \( t = 6 \) hours, we have that \( e^{-6k} = \frac{1}{2} \), or \(-6k = \ln \frac{1}{2} = -\ln 2 \), so \( k = \ln 2/6 \). This gives the differential equation for \( A \).

(b) The rate of change of the amount of \( B \) comes from two effects: \( B \) is being produced at the same rate as \( A \) is decaying, i.e., \((\ln 2/6)A\), and at the same time \( B \) is decaying at a rate of \(-\ln 2/9B\). Therefore

\[
\frac{dB}{dt} = \frac{\ln 2}{6} A - \frac{\ln 2}{9} B.
\]

(c) We’ve explained the first two equations in parts (a) and (b), and the third equation is explained by the fact that the rate of increase of the amount of \( C \) is equal to the rate of decrease of the amount of \( B \).
(d) To solve this system, we first set \( \alpha = \frac{1}{6} \ln 2 \) and \( \beta = \frac{1}{5} \ln 2 \) so we don’t have to deal with the numbers until the end. Then the problem becomes

\[
\begin{bmatrix}
A & B \\
B & C
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta
\end{bmatrix}
= \begin{bmatrix}
-\alpha & 0 & 0 \\
\alpha & -\beta & 0 \\
0 & \beta & 0
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\beta \\
\beta
\end{bmatrix}
\]

and we need to find the eigenvalues and eigenvectors of

\[
M = \begin{bmatrix}
-\alpha & 0 & 0 \\
\alpha & -\beta & 0 \\
0 & \beta & 0
\end{bmatrix}.
\]

Because of all the zeroes, it’s easy to see that

\[
\det(M - \lambda I) = (-\alpha - \lambda)(-\beta - \lambda)(-\lambda) = -\lambda(\lambda + \alpha)(\lambda + \beta),
\]

so the eigenvalues of \( M \) are 0, \(-\alpha\) and \(-\beta\).

For \( \lambda = 0 \) we need the nullspace of \( M \), which is spanned by the vector \( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \). For \( \lambda = -\alpha \), we need the nullspace of

\[
M + \alpha I = \begin{bmatrix}
0 & 0 & 0 \\
\alpha & -\beta + \alpha & 0 \\
0 & \beta & \alpha
\end{bmatrix}
\]

which is spanned by \( \begin{bmatrix} \beta - \alpha \\ \alpha \\ -\beta \end{bmatrix} \)

and for \( \lambda = -\beta \) we need the nullspace of

\[
M + \beta I = \begin{bmatrix}
-\alpha + \beta & 0 & 0 \\
\alpha & 0 & 0 \\
0 & \beta & \beta
\end{bmatrix}
\]

which is spanned by \( \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \)

So the general solution of the system of differential equations is given by

\[
\begin{bmatrix}
A \\
B \\
C
\end{bmatrix}
= c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-\alpha t} \begin{bmatrix} \beta - \alpha \\ \alpha \\ -\beta \end{bmatrix} + c_3 e^{-\beta t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.
\]

To match the initial conditions \( A(0) = A_0, B(0) = 0, C(0) = 0 \) we need

\[
c_2 = \frac{A_0}{\beta - \alpha}, \quad c_3 = \frac{\alpha A_0}{\beta - \alpha}, \quad \text{and} \quad c_3 = A_0.
\]

We conclude that the solution is

\[
A = A_0 e^{-\alpha t}
\]

\[
B = \frac{\alpha A_0}{\beta - \alpha} (e^{-\alpha t} - e^{-\beta t})
\]

\[
C = A_0 + \frac{A_0}{\beta - \alpha} (\alpha e^{-\beta t} - \beta e^{-\alpha t})
\]

5. In class on Thursday, we started to think about what happens when we are trying to find the general solution of \( x' = Ax \) and \( A \) has an eigenvalue \( \lambda_0 \) that is a double root of \( \det(A - \lambda I) \) but
which has only one linearly independent eigenvector \( v_0 \) (i.e., the space of solutions of \( (A - \lambda_0 I)v = \mathbf{0} \) is only one-dimensional). So we know that \( x = ce^{\lambda_0 t}v_0 \) is a solution of the system of differential equations, but we need to find a second, independent solution and don’t have anything at hand to make it out of.

However, in this case, it is always true that there will be a vector \( w_0 \), linearly independent from \( v_0 \), that satisfies \( (A - \lambda_0 I)w_0 = v_0 \). It is standard to call \( w_0 \) a “generalized eigenvector” of \( A \) corresponding to the eigenvalue \( \lambda_0 \), since \((A - \lambda_0 I)^2w_0 = \mathbf{0}\).

(a) Prove this last statement. I.e., show that if \((A - \lambda_0 I)w_0 = v_0\), then \((A - \lambda_0 I)^2w_0 = \mathbf{0}\).

(b) Show that, in this case \( x = e^{\lambda_0 t}(tv_0 + w_0) \) is a linearly independent solution of \( x' = Ax \).

(c) Use part (b) to find the general solution of \( x' = Ax \) for \( A = \begin{bmatrix} 4 & -9 \\ 4 & -8 \end{bmatrix} \).

(a) Start with \((A - \lambda_0 I)w_0 = v_0\), and multiply both sides on the left by \((A - \lambda_0 I)\) to get:
\[
(A - \lambda_0 I)(A - \lambda_0 I)w_0 = (A - \lambda_0 I)v_0
\]
\[
(A - \lambda_0 I)^2w_0 = \mathbf{0}
\]
since we know that \((A - \lambda_0 I)v_0 = \mathbf{0}\) because \(v_0\) is an eigenvector of \( A \) with eigenvalue \( \lambda_0 \).

(b) First, we show that \( x \) is a solution of \( x' = Ax \): We compute
\[
x' = \lambda_0 e^{\lambda_0 t}(tv_0 + w_0) + e^{\lambda_0 t}v_0.
\]
Also (noting that because \((A - \lambda_0 I)w_0 = v_0\), we have \(Aw_0 = v_0 + \lambda_0 w_0\)):
\[
Ax' = e^{\lambda_0 t}(tAv_0 + Aw_0) = e^{\lambda_0 t}(t\lambda_0 v_0 + v_0 + \lambda_0 w_0)
\]
which is the same as \( x' \).

Next, we have to show that this \( x \) is linearly independent from the “standard” \( x = e^{\lambda_0 t}v_0 \). This follows from the fact that \( v_0 \) and \( w_0 \) must be linearly independent, since \( v_0 \) is in the nullspace of \( A - \lambda_0 I \) and \( w_0 \) isn’t.

(c) We use the eigenvalue/eigenvector method, and start the usual way:
\[
\det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & -9 \\ 4 & -8 - \lambda \end{bmatrix} = (4 - \lambda)(-8 - \lambda) + 36 = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2.
\]
So \( \lambda = -2 \) is the only eigenvalue. To find the corresponding eigenvector(s), we need the nullspace of
\[
A + 2I = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}
\]
which is spanned by \( v = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \).

Since there’s only one linearly independent eigenvector, we need (and will be able to find) a generalized eigenvector, i.e., to solve \((A - \lambda I)w = v\) for \( w \):
\[
\begin{bmatrix} 6 & -9 & 3 \\ 4 & -6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & 0 \end{bmatrix}
\]
so we have

\[ w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}. \]

The formula in part (b) tells us that the general solution of the differential system is

\[ x = c_1 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-2t} \left( t \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = c_1 e^{-2t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 3t + 2 \\ 2t + 1 \end{bmatrix} \]