MATH 240 – Homework assignment 11 – April 7, 2015

Make sure you can answer all of the True-False review questions at the end of sections 7.5, 7.8, 7.9 and 7.10 of the textbook. Also, make sure you can do the “core problems” (section 7.5: 7, 15, 16, 17; section 7.8: 3, 6, 10, 11; section 7.9: 11, 17, 20, 23; and section 7.10: 1, 6, 9).

Then write up solutions to the following to hand in on Tuesday April 14:

1. Practice, practice. Find the general solution of the system $x' = Ax$ if

   (a) $A = \begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix}$.

   (b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$

   (d) $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$

   (e) For parts (a) and (c) above, draw some representative graphs of solutions in the phase plane.

(a) The characteristic polynomial of $A$ is

$$\det(A - \lambda I) = \det \begin{bmatrix} 3 - \lambda & 5 \\ -1 & -1 - \lambda \end{bmatrix} = (3 - \lambda)(-1 - \lambda) + 5 = \lambda^2 - 2\lambda + 2.$$ 

The roots of this equation are $\lambda = \frac{2 \pm \sqrt{1 - 8}}{2} = 1 \pm i$, so the eigenvalues are $1 \pm i$. We only need to concentrate on one of these, so we choose $\lambda = 1 + i$, and so we find the kernel of:

$$\begin{bmatrix} 2 - i \\ -1 \end{bmatrix} \text{ which is spanned by } \begin{bmatrix} 2 + i \\ -1 \end{bmatrix}.$$ 

Therefore, we get two linearly independent solutions of $x' = Ax$ by taking the real and imaginary parts of

$$e^{(1+i)t} \begin{bmatrix} 2 + i \\ -1 \end{bmatrix} = e^t(\cos t + i \sin t) \begin{bmatrix} 2 + i \\ -1 \end{bmatrix} = \begin{bmatrix} e^t(2\cos t - \sin t) + ie^t(\cos t + 2\sin t) \\ -e^t \cos t - ie^t \sin 2t \end{bmatrix}.$$ 

The general solution of the $x' = Ax$ is thus

$$x = c_1 e^t \begin{bmatrix} 2\cos t - \sin t \\ -\cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t + 2\sin t \\ -\sin t \end{bmatrix}.$$ 

(b) The characteristic polynomial of $A$ is

$$\det(A - \lambda I) = \det \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix} = (1 - \lambda)^3 + 4(1 - \lambda) = (1 - \lambda)(\lambda^2 - 2\lambda + 5)$$
so the eigenvalues of $A$ are $\lambda = 1, \lambda = 1 \pm 2i$. For $\lambda = 1$ we have to find the nullspace of:

$$
\begin{bmatrix}
0 & 0 & 0 \\
2 & 0 & -2 \\
3 & 2 & 0
\end{bmatrix}
$$

which is spanned by

$$
\begin{bmatrix}
2 \\
-3 \\
2
\end{bmatrix}
$$

and, since we only have to deal with one of the two complex conjugate eigenvalues, for $\lambda = 1 + 2i$ we have to find the nullspace of

$$
\begin{bmatrix}
-2i & 0 & 0 \\
2 & -2i & -2 \\
3 & 2 & -2i
\end{bmatrix}
$$

which is spanned by

$$
\begin{bmatrix}
0 \\
1 \\
-i
\end{bmatrix}
$$

For this, we need to calculate the real and imaginary parts of

$$
e^{1+2i} \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = e^t (\cos 2t + i \sin 2t) \begin{bmatrix} 0 \\ 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^t \cos 2t \\ e^t \sin 2t \\ -e^t \cos 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}.
$$

We conclude that the general solution is

$$
\mathbf{x} = \begin{bmatrix} 2c_1 e^t \\ -3c_1 e^t \\ 2c_1 e^t \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 e^t \cos 2t \\ c_2 e^t \sin 2t \end{bmatrix} + \begin{bmatrix} 0 \\ c_3 e^t \sin 2t \\ -c_3 e^t \cos 2t \end{bmatrix}.
$$

(c) The characteristic polynomial of $A$ is

$$
det(A - \lambda I) = det \begin{bmatrix} 1 - \lambda & -2 \\ 2 & -3 - \lambda \end{bmatrix} = (1 - \lambda)(-3 - \lambda) + 4 = \lambda^2 + 2\lambda + 1 = (\lambda + 1)^2
$$

so the only eigenvalue of $A$ is $\lambda = -1$. It has only one linearly independent eigenvector, which spans the nullspace of

$$
\begin{bmatrix}
2 & -2 \\
2 & -2
\end{bmatrix}
$$

which is spanned by

$$
\mathbf{v} = \begin{bmatrix}
1 \\
1
\end{bmatrix}.
$$

We will need a generalized eigenvector $\mathbf{w}$, i.e., one for which $(A + I)\mathbf{w} = \mathbf{v}$. It’s easy to see that there are lots of possibilities, but we can take $\mathbf{w} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$. Then the general solution of the system of differential equations is

$$
\mathbf{x} = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} c_1 e^{-t} + c_2 te^{-t} + \frac{1}{2}c_2 e^{-t} \\ c_1 e^{-t} + c_2 te^{-t} \end{bmatrix}.
$$

(d) The characteristic polynomial of $A$ is

$$
det(A - \lambda I) = det \begin{bmatrix} 2 - \lambda & 1 & -1 \\ 0 & -1 - \lambda & 2 \\ 0 & 0 & -1 - \lambda \end{bmatrix} = (2 - \lambda)(1 + \lambda)^2
$$

so the eigenvalues of $A$ are $\lambda = 2$ and $\lambda = -1$ (twice). For $\lambda = 2$ we have to find the nullspace of

$$
\begin{bmatrix}
0 & 1 & -1 \\
0 & -3 & 2 \\
0 & 0 & -3
\end{bmatrix}
$$

which is spanned by

$$
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}.$$
The double eigenvalue $\lambda = -1$ has only one linearly independent eigenvector, which spans the nullspace of
\[
\begin{bmatrix}
3 & 1 & -1 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{bmatrix}
\]
which is spanned by $v = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}$.

We will need a generalized eigenvector $w$, i.e., one for which $(A + I)w = v$. The third component of such a vector must be equal to $-3/2$, and then we can choose the second component to be $-1/2$ and the first component to be zero, so
\[
w = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}.
\]
(There are many other generalized eigenvectors, obtained from this one by adding multiples of $v$.)

Then we can write the general solution of the system of differential equations as:
\[
x = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + c_3 e^{-t} \left( \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ -\frac{3}{2} \end{bmatrix} \right) = \begin{bmatrix} c_1 e^{2t} + c_2 e^{-t} + c_3 t e^{-t} \\ -3c_2 e^{-t} - 3c_3 t e^{-t} - \frac{3}{2} c_3 e^{-t} \\ -\frac{3}{2} c_3 e^{-t} \end{bmatrix}
\]

(e) Now we need some pictures. Here is the phase plane with the direction field and some representative spiral curves drawn. The curves spiral outward from the origin:

![Phase Plane](image)

Figure 1: Solutions of $x' = 3x + 5y$, $y' = -x - y$. 
2. Solve the linear system

\[
\begin{align*}
x' &= -x + y \\
y' &= cx - y
\end{align*}
\]

in terms of \(c\), and draw some representative curves for each of \(c = \frac{1}{4}, 4, 0, -9\).

Let \(A = \begin{bmatrix} -1 & 1 \\ c & -1 \end{bmatrix}\) be the matrix of the system. For all \(c\), the trace of \(A\) is \(-2\) and the determinant of \(A\) is \(1 - c\). So we know from our geometric analysis of 2-by-2 linear systems of differential equations that for \(c > 1\) the origin is a saddle point (\(\det A < 0\)), for \(0 < c \leq 1\) the origin is a stable node (a “sink”, because for these values of \(c\) we have \((\text{tr}(A))^2 - 4\det(A) \geq 0\)), and for \(c < 0\) we have a stable spiral (inward-pointing spiral).

Now that we know what to expect, we calculate. The characteristic polynomial of \(A\) is

\[
\det(A - \lambda I) = \det\begin{bmatrix} -1 - \lambda & 1 \\ c & -1 - \lambda \end{bmatrix} = (-1 - \lambda)^2 - c = \lambda^2 + 2\lambda + 1 - c,
\]

so the eigenvalues of \(A\) are \(-1 \pm \sqrt{c}\).

If \(c > 0\) then we have real eigenvalues and proceed as follows: For \(\lambda = -1 + \sqrt{c}\) we have to find the nullspace of

\[
\begin{bmatrix} 1 \\ \sqrt{c} \end{bmatrix}
\]

which is spanned by \(\begin{bmatrix} 1 \\ \sqrt{c} \end{bmatrix}\).

And for \(\lambda = -1 - \sqrt{c}\) we have to find the nullspace of

\[
\begin{bmatrix} -1 \\ \sqrt{c} \end{bmatrix}
\]

which is spanned by \(\begin{bmatrix} -1 \\ \sqrt{c} \end{bmatrix}\).

So the general solution when \(c > 0\) is given by

\[
\begin{bmatrix} x \\ y \end{bmatrix} = c_1 e^{(-1 + \sqrt{c})t} \begin{bmatrix} 1 \\ \sqrt{c} \end{bmatrix} + c_2 e^{(-1 - \sqrt{c})t} \begin{bmatrix} 1 \\ -\sqrt{c} \end{bmatrix} = \begin{bmatrix} c_1 e^{(-1 + \sqrt{c})t} + c_2 e^{(-1 - \sqrt{c})t} \\ c_1 \sqrt{c} e^{(-1 + \sqrt{c})t} - c_2 \sqrt{c} e^{(-1 - \sqrt{c})t} \end{bmatrix}.
\]
If \( c = 0 \) then \( \lambda = -1 \) is a double eigenvalue, and its eigenspace is the nullspace of

\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\]

which is spanned by \( v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \).

Since there is only one linearly independent eigenvector, we need a generalized eigenvector \( w \), i.e., on \( e \) for which \((A + I)w = v\). The simplest such \( w \) (to which we could add any multiple of \( v \)) is \( w = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \), so the general solution in this case is

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t} \left( t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} c_1 e^{-t} + c_2 t e^{-t} \\ c_2 e^{-t} \end{bmatrix}.
\]

Finally, if \( c < 0 \) then the eigenvalues are complex, and we can write them as \( \lambda = -1 \pm i\sqrt{-c} \) (since \( -c \) is positive). We only need to consider one of the eigenvalues in this case, so we use \( \lambda = -1 + i\sqrt{-c} \) and find the nullspace of

\[
\begin{bmatrix}
-\frac{i}{\sqrt{-c}} & 1 \\
\frac{c}{i\sqrt{-c}} & -i\sqrt{-c}
\end{bmatrix}
\]

which is spanned by \( \begin{bmatrix} 1 \\ i\sqrt{-c} \end{bmatrix} \).

Therefore, we get two linearly independent solutions of \( x' = Ax \) by taking the real and imaginary parts of

\[
e^{(-1+i\sqrt{-c})t} \begin{bmatrix}
1 \\
i\sqrt{-c}
\end{bmatrix} = e^{-t} \left( \cos(\sqrt{-c}t) + i\sin(\sqrt{-c}t) \right) \begin{bmatrix} 1 \\ i\sqrt{-c} \end{bmatrix}
\]

\[
= \begin{bmatrix}
e^{-t} \cos(\sqrt{-c}t) \\
-\sqrt{-c} e^{-t} \sin(\sqrt{-c}t)
\end{bmatrix} + i \begin{bmatrix}
e^{-t} \sin(\sqrt{-c}t) \\
\sqrt{-c} e^{-t} \cos(\sqrt{-c}t)
\end{bmatrix}
\]

The general solution of \( x' = Ax \) in this case is thus

\[
x = c_1 e^{-t} \begin{bmatrix} \cos(\sqrt{-c}t) \\ -\sqrt{-c} \sin(\sqrt{-c}t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin(\sqrt{-c}t) \\ \sqrt{-c} \cos(\sqrt{-c}t) \end{bmatrix}
\]

For \( c = \frac{1}{4} \), we have that the general solution is

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = c_1 e^{-\frac{t}{4}} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + c_2 e^{-\frac{t}{4}} \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}
\]

and some representative trajectories are shown here:
Figure 3: Solutions of $x' = -x + y$, $y' = \frac{1}{4}x - y$

Here is the picture for $c = 0$:

Figure 4: Solutions of $x' = -x + y$, $y' = -y$
And here is the picture for $c = -9$ (inward spirals): and for $c = 4$:

Figure 5: Solutions of $x' = -x + y$, $y' = -9x - y$

Figure 6: Solutions of $x' = -x + y$, $y' = 4x - y$

3. Find and analyze the critical points of the system

$$x' = y, \quad y' = x^3 - x.$$ 

There’s one at the origin — to analyze it, find the equation for $dy/dx$ implied by the system.

The critical points of the system occur where $y = 0$ and $x^3 - x = x(x - 1)(x + 1) = 0$, so there are three of them: $(-1, 0)$, $(0, 0)$ and $(1, 0)$. The Jacobian of the system is

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & 0 \end{bmatrix}.$$
For both the critical points at \((-1, 0)\) and \((1, 0)\), the Jacobian matrix is

\[
J(\pm 1, 0) = \begin{bmatrix}
0 & 1 \\
2 & 0
\end{bmatrix}.
\]

Since the determinant of this matrix is negative, each of these is a saddle point.

For the point \((0, 0)\) we have that

\[
J(0, 0) = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]

which has determinant 1 and trace 0, so it is on the borderline between being an inward and an outward spiral. To determine whether the point is a spiral point or a center (surrounded by closed curves), we use the chain rule to obtain a differential equation that will have the curves in the phase plane as solutions by eliminating \(t\), as follows. The chain rule says that

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}, \quad \text{therefore} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}
\]

We use this together with the original differential system to write

\[
\frac{dy}{dx} = \frac{x^3 - x}{y},
\]

which is a separable first-order equation with solutions

\[
y^2 = \frac{1}{2} x^4 - x^2 + C.
\]

If \(C\) is small and positive, the graph of this equation includes a closed curve surrounding the origin. We conclude that the critical point \((0, 0)\) is a center (or vortex), surrounded by closed curves.

With the two saddle points and the center, we conclude that the phase portrait for this system of differential equations looks as follows:
4. Consider the nonlinear differential equation $x'' + x = 1 + \varepsilon x^2$. Convert this to a system of first-order equations, and then find and classify the critical points of the system.

If we let $y = x'$ then the linear system is

$$
\begin{align*}
x' &= y \\
y' &= 1 - x + \varepsilon x^2
\end{align*}
$$

so for $\varepsilon \neq 0$ the critical points occur where $y = 0$ and $1 - x + \varepsilon x^2 = 0$, i.e., where $x = \frac{1 \pm \sqrt{1 - 4\varepsilon}}{2\varepsilon}$ (notice that $a = \varepsilon$ in the quadratic formula). So there are two critical points, unless $\varepsilon = 0$ or $\varepsilon \geq \frac{1}{4}$, since when $\varepsilon = 0$ the system is linear and there is a single critical point at $(1, 0)$, and when $\varepsilon = \frac{1}{4}$ the two critical points coincide (at $x = 2$ and $y = 0$), and for $\varepsilon > \frac{1}{4}$ there are no real critical points.

To analyze the critical points, we will of course need the Jacobian matrix

$$
J(x, y) = \begin{bmatrix} 0 & 1 \\ 2\varepsilon x - 1 & 0 \end{bmatrix}
$$

which has zero trace and determinant $1 - 2\varepsilon x$.

The case where $\varepsilon = 0$ is easy to understand, since the critical point $(1, 0)$ in this case will be a center, surrounded by actual circles. So we turn to the case where $\varepsilon \neq 0$.

First, consider the critical point at $x = \frac{1 - \sqrt{1 - 4\varepsilon}}{2\varepsilon}, y = 0$. For $\varepsilon$ near zero, this critical point is on the $x$-axis near $(1, 0)$ (as $\varepsilon \to -\infty$ this critical point tends toward the origin along the positive $x$-axis, and as $\varepsilon \to \frac{1}{4}$ from below this critical point approaches $(2, 0)$ along the $x$-axis from the left. The Jacobian at this point is

$$
J\left(\frac{1 - \sqrt{1 - 4\varepsilon}}{2\varepsilon}, 0\right) = \begin{bmatrix} 0 & 1 \\ -\sqrt{1 - 4\varepsilon} & 0 \end{bmatrix},
$$

which has zero trace and positive determinant, so we’re in the borderline case between inward and
outward spirals again. From the system of differential equations we have
\[ \frac{dy}{dx} = \frac{1 - x + \varepsilon x^2}{y} \]
which has solutions
\[ y^2 = 2x - x^2 + \frac{2}{3}\varepsilon x^3 + C. \]
So this critical point will be surrounded by closed curves (it being a local minimum of \( y^2 - 2x + x^2 + \frac{2}{3}\varepsilon x^3 + C \)).

The other critical point, at \( x = \frac{1 + \sqrt{1 - 4\varepsilon^2}}{2\varepsilon}, y = 0 \), is easier to analyze. We have
\[ J\left( \frac{1 + \sqrt{1 - 4\varepsilon^2}}{2\varepsilon}, 0 \right) = \begin{bmatrix} 0 & 1 \\ \sqrt{1 - 4\varepsilon^2} & 0 \end{bmatrix}, \]
which has negative determinant, so this is a saddle point.

Here is the phase portrait when \( \varepsilon = -0.15 \) (so the saddle point is on the negative \( x \)-axis):

![Phase Portrait](image)

Figure 8: Solutions of \( x' = y, y' = 1 - x - 0.15x^2 \)
Here is the phase portrait when $\varepsilon = +0.15$ (so the saddle point is on the positive $x$-axis):

Figure 9: Solutions of $x' = y$, $y' = 1 - x + 0.15x^2$

And here is the phase portrait when $\varepsilon = -0.3$ (so both the saddle point and the center have disappeared):

Figure 10: Solutions of $x' = y$, $y' = 1 - x + 0.3x^2$