MATH 240 – Homework assignment 1 – January 15, 2015

Make sure you can answer all of the True-False review questions at the end of sections 2.1 and 2.2 of the textbook. Also, make sure you can do the “core problems” (section 2.1: 9, 11, 23, 25 and section 2.2: 5, 13, 16, 19, 39, 43).

Then write up solutions to the following to hand in on Tuesday January 20:

1. Find formulas for \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n \) and \( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n \).

Try a few cases:

\( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \)

and

\( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \).

So it would appear that

\( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \).

To prove this, note that we already know that it’s true for \( n = 1, 2, 3 \). So we’ll show that if it’s true for \( n \) then it is also true for \( n + 1 \), which will complete the proof by mathematical induction. So assuming the formula is true for \( n \), we calculate

\( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{n+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n+1 \\ 0 & 1 \end{bmatrix} \)

and we are done.

Likewise

\( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \)

and

\( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \).

The tricky part seems to be the pattern of the numbers in the first row, third column. But 1,3,6 are the first three “triangular” numbers (1 = 1, 3 = 1 + 2, 6 = 1 + 2 + 3 etc), and the formula for the \( n \)th triangular number is \( \frac{1}{2}n(n+1) \). So it appears that

\( \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & \frac{1}{2}n(n+1) \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \).
To prove this, we once again know that it is true for \( n = 1, 2, 3 \). So assume that it’s true for \( n \) and we’ll see if it’s true for \( n + 1 \) as follows:

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}^{n+1} = \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}^n \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & n & \frac{1}{2}n(n+1) \\
0 & 1 & n \\
0 & 0 & 1 \\
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix} \\
= \begin{bmatrix}
1 & n+1 & 1 + n + \frac{1}{2}n(n+1) \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
1 & n+1 & \frac{1}{2}(n+1)(n+2) \\
0 & 1 & 1 \\
0 & 0 & 1 \\
\end{bmatrix}
\]

which is what we needed to show.

2. (a) Explain why (prove that), for square matrices \( A \) and \( B \), it is true that \( \text{tr}(AB) = \text{tr}(BA) \).

(b) Recall that the \textit{commutator} of two square matrices \( A \) and \( B \) is the matrix \( [A, B] = AB - BA \). Explain why it is impossible for \( [A, B] = I \), where \( I \) is the appropriately-sized identity matrix.

(a) The trace of a matrix is the sum of its diagonal elements. The \( i \)th diagonal element of \( AB \) is the sum of the products of the elements of the \( i \)th row of \( A \) with the corresponding elements of the \( i \)th column of \( B \), in other words, the sum of all products of the form \( a_{ik}b_{ki} \) for all values of \( k \). Since the trace adds these all together for all values of \( i \), we have that \( \text{tr}(AB) \) is the sum of all products of the form \( a_{ik}b_{ki} \) over all \( i \) and \( k \). Clearly the trace of \( BA \) is the same sum (just organized differently), so \( \text{tr}(AB) = \text{tr}(BA) \). That is the reason the equality holds. In symbols, which may or may not be clearer:

\[
\text{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} \left( \sum_{k=1}^{n} a_{ik}b_{ki} \right) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik}b_{ki} \\
= \sum_{i=1}^{n} \sum_{k=1}^{n} b_{ki}a_{ik} = \sum_{k=1}^{n} \sum_{i=1}^{n} b_{ki}a_{ik} \\
= \sum_{k=1}^{n} \left( \sum_{i=1}^{n} b_{ki}a_{ik} \right) = \sum_{k=1}^{n} (BA)_{kk} \\
= \text{tr}(BA)
\]

(b) Take the trace of both sides of the equation \( [A, B] = I \). The easy side first: \( \text{tr}(I) = n \) if the matrices are \( n \)-by-\( n \), but

\[
\text{tr}([A, B]) = \text{tr}(AB - BA) = \text{tr}(AB) - \text{tr}(BA) = 0
\]

by part (a). So if the \( [A, B] \) were equal to \( I \) we would have the equation \( 0 = n \), which is impossible.

3. Let \( M(t) \) be the matrix function

\[
M(t) = \begin{bmatrix}
e^t & 3e^{2t} - 3e^t \\
0 & e^{2t}
\end{bmatrix}
\]
Show that $M(t)$ satisfies the matrix differential equation:

$$\frac{dM}{dt} = AM,$$

where $A$ is the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}.$$

We simply calculate both sides of the differential equation and see whether they are the same:

$$\frac{dM}{dt} = \frac{d}{dt} \begin{bmatrix} e^t & 3e^{2t} - 3e^t \\ 0 & e^{2t} \end{bmatrix} = \begin{bmatrix} e^t & 6e^{2t} - 3e^t \\ 0 & 2e^{2t} \end{bmatrix}$$

and

$$AM = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} e^t & 3e^{2t} - 3e^t \\ 0 & e^{2t} \end{bmatrix} = \begin{bmatrix} e^t & 6e^{2t} - 3e^t \\ 0 & 2e^{2t} \end{bmatrix}$$

so they’re the same all right.