MATH 240 – Homework assignment 4 – February 3, 2015

Make sure you can answer all of the True-False review questions at the end of sections 4.4, 4.6, 4.7 and 4.9 of the textbook. Also, make sure you can do the “core problems” (section 4.4: 7, 9, 12, 15, 17, 19, 25, 27, 28; section 4.6: 6, 11, 15, 17, 23, 25, 28; section 4.7: 5, 19, 23, 27, 38; section 4.9: 6, 9, 13, 15, 18).

Then write up solutions to the following to hand in on Tuesday February 10:

1. Suppose $S$ and $T$ are subspaces of a vector space $V$.
   (a) Show that the (set-theoretic) intersection $S \cap T$ is also a subspace of $V$.
   (b) Show that the (set-theoretic) union $S \cup T$ is not in general a subspace of $V$ (by giving examples where this is not the case).
   (c) Define the set $S + T$ via:
   $$S + T = \{x + y \mid x \in S \text{ and } y \in T\}$$
in other words any element of $S + T$ can be written as the sum of two vectors, one from $S$ and one from $T$. Show that $S + T$ is a subspace of $V$.
   (d) Show that
   $$\dim(S + T) = \dim S + \dim T - \dim(S \cap T).$$
   (The trick is to choose a “clever” basis for $S + T$).

2. Compute the dimension and find bases for the following vector spaces:
   (a) Real skew-symmetric 4-by-4 matrices.
   (b) Polynomials $p(x)$ of degree 4 which have the property that $p(2) = 0$ and $p(3) = 0$.
   (c) Cubic polynomials $p(x, y)$ in two real variables with the properties: $p(0, 0) = 0$, $p(1, 0) = 0$ and $p(0, 1) = 0$.

3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Show that the set of 2-by-2 matrices that commute with $A$ (i.e., matrices $B$ for which $AB = BA$) is a subspace of the vector space of 2-by-2 matrices, and find the dimension of and a basis for this subspace.

4. (a) Show that the set of polynomials $B = \{1, x, x(x-1), x(x-1)(x-2), x(x-1)(x-2)(x-3)\}$ is a basis for the vector space $P_4$ of quartic polynomials.
   (b) By multiplying out the elements of $B$, you can express them in terms of the standard basis for $P_3$, namely $A = \{1, x, x^2, x^3, x^4\}$. Explain how to use linear algebra (in particular, the computation of a certain inverse matrix) to express the elements of $A$ in terms of the elements of $B$. We’ll talk more about this and why it’s useful next week.