MATH 240 – Homework assignment 4 – February 3, 2015

Make sure you can answer all of the True-False review questions at the end of sections 4.4, 4.6, 4.7 and 4.9 of the textbook. Also, make sure you can do the “core problems” (section 4.4: 7, 9, 12, 15, 17, 19, 25, 27, 28; section 4.6: 6, 11, 15, 17, 23, 25, 28; section 4.7: 5, 19, 23, 27, 38; section 4.9: 6, 9, 13, 15, 18).

Then write up solutions to the following to hand in on Tuesday February 10:

1. Suppose $S$ and $T$ are subspaces of a vector space $V$.

   (a) Show that the (set-theoretic) intersection $S \cap T$ is also a subspace of $V$.

   (b) Show that the (set-theoretic) union $S \cup T$ is not in general a subspace of $V$ (by giving examples where this is not the case).

   (c) Define the set $S + T$ via:
   \[ S + T = \{ x + y \mid x \in S \text{ and } y \in T \} \]
   in other words any element of $S + T$ can be written as the sum of two vectors, one from $S$ and one from $T$. Show that $S + T$ is a subspace of $V$.

   (d) Show that
   \[ \dim(S + T) = \dim S + \dim T - \dim(S \cap T). \]
   (The trick is to choose a “clever” basis for $S + T$).

   (a) We have to check two things: if $v$ and $w$ are arbitrary vectors in $S \cap T$ then $v + w$ is also in $S \cap T$ and for any scalar $s$, that $sv$ is in $S \cap T$. So first, suppose first that $v$ and $w$ are in $S \cap T$. This means that $v$ and $w$ are both in $S$ and that $v$ and $w$ are both in $T$. Since $S$ and $T$ are themselves subspaces of $V$, this implies that $v + w$ is in $S$ and that $v + w$ is in $T$, and so $v + w \in S \cap T$. Likewise, $sv$ is in $S$ and $sv$ is in $T$, and so $sv$ is in $S \cap T$. So $S \cap T$ is also a subspace of $V$.

   (b) Here is an example: the set $S$ of vectors of the form $[a_1, 0]$ is a subspace of $\mathbb{R}^2$, as is the set $T$ of vectors of the form $[0, a_2]$. But then the set $S \cup T$ contains only vectors where one or the other (or both) of the components is zero, and so it does not contain the vector $[1, 1] = [1, 0] + [0, 1]$, even though both $[1, 0]$ and $[0, 1]$ are in $S \cup T$. Therefore $S \cup T$ is not a subspace of $\mathbb{R}^2$.

   (c) Suppose $v$ and $w$ are in $S + T$. Then $v = s_1 + t_1$ and $w = s_2 + t_2$ for vectors $s_1$ and $s_2$ in $S$ and vectors $t_1$ and $t_2$ in $T$. But then $v + w = (s_1 + t_1) + (s_2 + t_2) = (s_1 + s_2) + (t_1 + t_2)$, and, since $S$ and $T$ are subspaces, $s_1 + s_2 \in S$ and $t_1 + t_2 \in T$, and so $v + w \in S + T$. Likewise, for any scalar $a$, $av = a(s_1 + t_1) = as_1 + at_1$, and $as_1 \in S$ and $at_1 \in T$, again because $S$ and $T$ are subspaces, and so $av \in S + T$. Thus $S + T$ is a subspace of $V$.

   (d) We choose a basis of $S + T$ as follows. Begin with the subspace $S \cap T$ of $S + T$, and let $\{v_1, \ldots, v_p\}$ be a basis of $S \cap T$, where $p = \dim(S \cap T)$.

   Since $S \cap T$ is a subspace of $S$, we have that $v_1, \ldots, v_p$ are linearly dependent vectors in $S$, so the set $\{v_1, \ldots, v_p\}$ can be completed to a basis $\{v_1, \ldots, v_p, s_{p+1}, \ldots, s_q\}$ of $S$, where $q = \dim S$. None of the $q - p$ vectors $s_{p+1}, \ldots, s_q$ are in $T$, because they are not in $S \cap T$. 

Likewise, since $S \cap T$ is a subspace of $T$, we have that $v_1, \ldots, v_p$ are linearly dependent vectors in $T$, so the set \{ $v_1, \ldots, v_p$ \} can be completed to a basis \{ $v_1, \ldots, v_p, t_{p+1}, \ldots, t_r$ \} of $T$, where $r = \dim T$. And none of the $r - p$ vectors $t_{p+1}, \ldots, t_r$ are in $S$, because they are not in $S \cap T$.

Altogether, the set \{ $v_1, \ldots, v_p, s_{p+1}, \ldots, t_r$ \} form a basis of $S + T$, since they are linearly independent and clearly span $S + T$. And so the dimension of $S + T$ is number of vectors in this basis, namely $p + (q - p) + (r - p) = q + r - p = \dim S + \dim T - \dim(S \cap T)$, which is what we set out to show.

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2. Compute the dimension and find bases for the following vector spaces:

(a) Real skew-symmetric 4-by-4 matrices.

(b) Polynomials $p(x)$ of degree 4 which have the property that $p(2) = 0$ and $p(3) = 0$.

(c) Cubic polynomials $p(x, y)$ in two real variables with the properties: $p(0, 0) = 0$, $p(1, 0) = 0$ and $p(0, 1) = 0$.

(a) The dimension of this space is 6 and a basis is

$$\begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix} \}$$

(b) One way to do this would be to start with all quartic (degree-4) polynomials written in the form $a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$, and imposing the equations $a_1 + 2a_3 + 4a_2 + 8a_3 + 16a_4 = 0$ and $a_0 + 3a_1 + 9a_2 + 27a_3 + 81a_4 = 0$, but that looks like a messy computation. Rather, you can use the factor theorem and write polynomials $p$ for which $p(2) = p(3) = 0$ as $(x - 2)(x - 3)(a_0 + a_1 x + a_2 x^2)$. In this way it is clear that the space of such polynomials is three-dimensional and a basis is

$$\{(x - 2)(x - 3), x(x - 2)(x - 3), x^2(x - 2)(x - 3)\}.$$  

(c) This one is a little harder. A cubic polynomial $p(x, y)$ has the form:

$$p(x, y) = a_{00} + a_{10} x + a_{01} y + a_{20} x^2 + a_{11} xy + a_{02} y^2 + a_{30} x^3 + a_{21} x^2 y + a_{12} xy^2 + a_{03} y^3.$$  

Since $p(0, 0) = a_{00}$, the condition $p(0, 0) = 0$ simply says that $a_{00} = 0$. Next, $p(1, 0) = a_{00} + a_{10} + a_{20} + a_{30}$. Since $a_{00}$ is already zero, the new condition is $a_{10} + a_{20} + a_{30} = 0$. Likewise, the condition $p(0, 1) = 0$ means that $a_{01} + a_{02} + a_{03} = 0$. These are clearly three independent conditions on the ten variables $a_{ij}$, so the dimension of the space is 7 and a basis is

$$\{x^2 - x, xy, y^2 - y, x^3 - x, x^2 y, xy^2, y^3 - y\}.$$
3. Let \( A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \). Show that the set of 2-by-2 matrices that commute with \( A \) (i.e., matrices \( B \) for which \( AB = BA \)) is a subspace of the vector space of 2-by-2 matrices, and find the dimension of and a basis for this subspace.

To show that the set of matrices that commute with \( A \) is a subspace, first note that the 2-by-2 zero matrix \( 0 \) commutes with \( A \), since \( A0 = 0A = 0 \) (this shows that the set of matrices that commute with \( A \) is not empty). Next, suppose that \( B \) and \( C \) are 2-by-2 matrices that commute with \( A \), so that \( AB = BA \) and \( AC = CA \). Then \( A(B + C) = AB + AC = BA + CA = (B + C)A \), so \( B + C \) also commutes with \( A \). Likewise, if \( s \) is a scalar, then \( A(sB) = s(AB) = s(BA) = (sB)A \), so \( sB \) also commutes with \( A \). Therefore the set of matrices that commute with \( A \) is closed under both addition and scalar multiplication and so is a subspace of the space of all 2-by-2 matrices.

Two (linearly independent, unless \( A = I \)) matrices that commute with \( A \) are obviously the identity matrix \( I \) and \( A \) itself. So the question is whether there are others besides linear combinations of \( I \) and \( A \).

To find out, begin by writing the matrix \( B \) as \( B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \). Then
\[
AB = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + 2b_{21} & b_{12} + 2b_{22} \\ 2b_{11} + b_{21} & 2b_{12} + b_{22} \end{bmatrix}
\]
and
\[
BA = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} b_{11} + 2b_{12} & 2b_{11} + b_{12} \\ b_{21} + 2b_{22} & 2b_{21} + b_{22} \end{bmatrix}
\]
Therefore
\[
AB - BA = \begin{bmatrix} 2b_{21} - 2b_{12} & 2b_{22} - 2b_{11} \\ 2b_{11} - 2b_{22} & 2b_{12} - 2b_{21} \end{bmatrix}
\]
At first glance, this seems to place four conditions on the matrix \( B \) (all four components of the matrix \( AB - BA \) must equal zero), but there are actually only two independent conditions, namely \( b_{21} = b_{12} \) and \( b_{11} = b_{22} \). So the matrices that commute with \( A \) are all of the form:
\[
\begin{bmatrix} p & q \\ q & p \end{bmatrix}
\]
for various values of \( p \) and \( q \). So a basis is
\[
\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}
\]
and the dimension of this space is 2 (note that we could also have used \( \{I, A\} \) as a basis.

4. (a) Show that the set of polynomials \( B = \{1, x, x(x - 1), x(x - 1)(x - 2), x(x - 1)(x - 2)(x - 3)\} \) is a basis for the vector space \( P_3 \) of quartic polynomials.

(b) By multiplying out the elements of \( B \), you can express them in terms of the standard basis for \( P_3 \), namely \( A = \{1, x, x^2, x^3, x^4\} \). Explain how to use linear algebra (in particular, the computation
of a certain inverse matrix) to express the elements of $A$ in terms of the elements of $B$. We’ll talk more about this and why it’s useful next week.

(a) Call the polynomials $p_0 = 1$, $p_1 = x$, $p_2 = x(x - 1)$, $p_3 = x(x - 1)(x - 2)$ and $p_4 = x(x - 1)(x - 2)(x - 3)$. There are five of them, and the dimension of $P_4$ is five, so we need only show that they are linearly independent. So suppose we have five constants $a_0, \ldots, a_4$ so that $a_0p_0 + a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4 = 0$. Since $p_4$ is the only one of the $p_i$’s to have an $x^4$ term, we must then have $a_4 = 0$ or else the sum will have a non-zero $x^4$ term. But if $x_4 = 0$, then $p_3$ is the only one of the remaining $p_i$’s to have a cubic term, so we must have $a_3 = 0$ or else the sum will have a non-zero $x^3$ term. And now that $a_4$ and $a_3$ are zero, then $p_2$ is the only one of the remaining $p_i$’s with a quadratic term, so $a_2$ must equal zero or else the sum will have a non-zero quadratic term. Likewise $a_1 = 0$ or else the sum will have a non-zero linear term, and then $a_0 = 0$ or else the sum will be a non-zero constant. So we’ve shown that the only way we can arrange for $a_0p_0 + a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4 = 0$ is to set $a_0 = a_1 = a_2 = a_3 = a_4 = 0$. Therefore the polynomials $p_0, \ldots, p_4$ are linearly independent and so $\{p_0, p_1, \ldots, p_4\}$ is a basis for $P_4$.

(b) We have $p_0 = 1$, $p_1 = x$, $p_2 = x^2 - x$, $p_3 = x^3 - 3x^2 + 2x$ and $p_4 = x^4 - 6x^3 + 11x^2 - 6x$. Therefore

$$a_0p_0 + a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4 + 4 = a_0 + (a_1 - a_2 + 2a_3 - 6a_4)x + (a_2 - 3a_3 + 11a_4)x^2 + (a_3 - 6a_4)x^3 + a_4x^4.$$

In matrix-speak, if $[a_0, a_1, a_2, a_3, a_4]_B$ is the vector of a polynomial in the basis $B$, then

$$\begin{bmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  c_4
\end{bmatrix}_S = \begin{bmatrix}
  1 & 0 & 0 & 0 & 0 \\
  0 & 1 & -1 & 2 & -6 \\
  0 & 0 & 1 & -3 & 11 \\
  0 & 0 & 0 & 1 & -6 \\
  0 & 0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4
\end{bmatrix}_B.$$

To go from the representation of a polynomial in the standard basis to the $B$ basis (i.e., how to represent $x^3$ as a linear combination of the $p_i$’s?), we need to invert the above matrix. This is not so hard, since the matrix is already upper-triangular with 1’s on the main diagonal (i.e., it is already in row-echelon form). So we go the rest of the way:

$$\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & -1 & 2 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & -3 & 11 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix} \begin{bmatrix}
  a_0 & 0 & 0 & 0 & 0 \\
  0 & a_1 & 0 & 0 & 0 \\
  0 & 0 & a_2 & 0 & 0 \\
  0 & 0 & 0 & a_3 & 0 \\
  0 & 0 & 0 & 0 & a_4
\end{bmatrix}_B.$$
So we have found the change of basis matrix $P_{B \rightarrow S}$ and we’ll have:

$$
\begin{bmatrix}
  a_0 \\
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
\end{bmatrix}_B =
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 \\
  0 & 1 & 1 & 1 & 1 \\
  0 & 0 & 1 & 3 & 7 \\
  0 & 0 & 0 & 1 & 6 \\
  0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
  c_0 \\
  c_1 \\
  c_2 \\
  c_3 \\
  c_4 \\
\end{bmatrix}_S.
$$

For instance, in the standard basis, the polynomial $x^3$ is represented by the vector $v_S = [0, 0, 0, 1, 0]^T_S$ (the transpose is to make $v_S$ a column vector). If we multiply this vector by $P_{B \rightarrow S}$ we should get $x^3$ represented as a sum of elements of the basis $B$. And $v_B = P_{B \rightarrow S} [0, 1, 3, 1, 0]^T_B$, and we can check:

$$
0(1)+1(x)+3(x(x-1))+1(x(x-1)(x-2))+0(x(x-1)(x-2)(x-3)) = x+(3x^2-3x)+(x^3-3x^2+2x) = x^3.
$$