MATH 240 – Homework assignment 7 – February 24, 2015

Make sure you can answer all of the True-False review questions at the end of sections 5.8 and 5.9 of the textbook. Also, make sure you can do the “core problems” (section 5.8: 7, 13, 17, 22, 23, 24, 32 and section 5.9: 7, 11, 15, 19).

Then write up solutions to the following to hand in on Tuesday March 3:

1. (Let \( A \) be an invertible matrix. If \( v \) is an eigenvector of \( A \), show it is also an eigenvector of both \( A^2 \) and \( A^{-2} \). What are the corresponding eigenvalues?)

2. Diagonalize the matrix
\[
A = \begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{bmatrix},
\]
by finding the eigenvalues of \( A \) listed in increasing order, corresponding eigenvectors, a diagonal matrix \( D \) and a matrix \( P \) such that \( A = PDP^{-1} \).

3. A certain \( 4 \times 4 \) matrix \( A \) has eigenvalues 0, 1, and 2. Suppose also that
\[
A \begin{bmatrix}
1 \\
0 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
2 \\
0 \\
2 \\
0
\end{bmatrix} \quad \text{and} \quad A \begin{bmatrix}
0 \\
1 \\
1 \\
0
\end{bmatrix} = \begin{bmatrix}
0 \\
2 \\
2 \\
0
\end{bmatrix}.
\]

(a) Determine whether there exist a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( A = P^{-1}DP \).

(b) If your answer to (a) is positive, find the sum of all entries of \( D \). Hint: this does not require finding \( D \) or \( P \).

4. Suppose that \( A \) is a \( 3 \times 3 \) matrix with eigenvalues \( \lambda_1 = 1 \), \( \lambda_2 = 0 \) and \( \lambda_3 = 1 \), and corresponding eigenvectors
\[
v_1 = \begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix}, \quad v_2 = \begin{bmatrix}
-1 \\
1 \\
0
\end{bmatrix}, \quad v_3 = \begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}.
\]

(a) Find the matrix \( A \).

(b) Find the matrix \( A^{20} \).

5. Let \( b \neq 0 \). Find the eigenvalues, eigenvectors, and determinant of \( A := \begin{bmatrix}
a & b & b & \cdots & b \\
b & a & b & \cdots & b \\
b & b & a & \cdots & b \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b & b & b & \cdots & a
\end{bmatrix} \).
6. Let \( b, c \neq 0 \). Find the eigenvalues, eigenvectors, and determinant of \( A := \begin{bmatrix} a & b & b & \cdots & b \\ c & a & 0 & \cdots & 0 \\ c & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c & 0 & 0 & \cdots & a \end{bmatrix} \).

7. (a) The matrix \( A \) is called a \textit{nilpotent} matrix if \( A^k = 0 \) for some \( k > 0 \). Show that an \( n \times n \) upper triangular matrix with zeroes on the diagonal is nilpotent.

(b) If \( A^2 = 0 \), what is \( e^A \)?

(c) Calculate \( e^A \) if \( A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \).

(d) The matrix \( P \) is called a \textit{projection} if \( P^2 = P \). Give an example of a 2-by-2 projection matrix that is not a diagonal matrix.

(e) Calculate \( e^{tP} \) for the matrix \( P \) you found in part (d). Then generalize and find a simple expression for \( e^{tP} \) for any projection matrix \( P \).

(f) The matrix \( R \) is called a \textit{reflection} if \( R^2 = I \). Give an example of a 2-by-2 reflection matrix that is not a diagonal matrix.

(g) Calculate \( e^{tR} \) for the matrix \( R \) you found in part (f). Then generalize and find a simple expression for \( e^{tR} \) for any reflection matrix \( R \).

(h) Calculate \( e^{tJ} \) for the matrix \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

(i) Look up the definitions of \( \cosh t \) and \( \sinh t \) if you have to. Then find a matrix \( A \) so that

\[
e^{tA} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}.
\]