1. Compute the determinants of the following matrices. The first one should be easy and each of the others should require little calculation, taking into account the one before.

\[
A = \begin{bmatrix}
2 & 1 & 0 & 0 \\
0 & 3 & 1 & 2 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[\text{det} A = \ldots\]

\[
B = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 3 & 1 & 2 \\
2 & 1 & 0 & 0
\end{bmatrix}
\]
\[\text{det} B = \ldots\]

\[
C = \begin{bmatrix}
0 & 3 & 1 & 3 \\
0 & 0 & 2 & 4 \\
0 & 3 & 1 & 2 \\
2 & 1 & 0 & 0
\end{bmatrix}
\]
\[\text{det} C = \ldots\]

\[
D = \begin{bmatrix}
2 & 4 & 1 & 3 \\
0 & 6 & 2 & 4 \\
-2 & -1 & 2 & 4 \\
2 & 1 & 0 & 0
\end{bmatrix}
\]
\[\text{det} D = \ldots\]

\[
E = \begin{bmatrix}
2 & 4 & 1 & 3 \\
-2 & -1 & 2 & 4 \\
0 & 6 & 2 & 4 \\
2 & 7 & 2 & 4
\end{bmatrix}
\]
\[\text{det} E = \ldots\]

Since \(A\) is upper triangular, its determinant is the product of the diagonal elements, so \(\text{det} A = 6\).

\(B\) obtained from \(A\) by switching rows 1 and 4 (changes sign of det) and then switching rows 2 and 3 (changes sign again) so \(\text{det} B = \text{det} A = 6\).

\(C\) is obtained from \(B\) by adding row 3 to row 1 (no change) and multiplying row 2 by 2 (multiplies det by 2), so \(\text{det} C = 2\text{det} B = 12\).

\(D\) is obtained from \(C\) by adding row 4 to row 1 (no change), exchanging rows 2 and 3 (change sign), and then doubling row 2 (multiply det by 2), so \(\text{det} D = -2\text{det} C = -24\).

\(E\) is obtained from \(D\) by adding row 2 to row 4 (no change) and then swapping rows 2 and 3 (changes sign of det), so \(\text{det} E = -\text{det} D = 24\).
1. Compute the determinants of the following matrices. The first one should be easy and each of the others should require little calculation, taking into account the one before.

\[
\begin{bmatrix}
1 & 1 & 3 & 0 \\
0 & 2 & 1 & 2 \\
0 & 0 & 4 & 2 \\
0 & 0 & 0 & 2
\end{bmatrix}
\quad \text{det } A = \ldots
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 2 \\
0 & 6 & 3 & 6 \\
0 & 0 & 4 & 2 \\
1 & 1 & 3 & 0
\end{bmatrix}
\quad \text{det } B = \ldots
\]

\[
\begin{bmatrix}
0 & 6 & 3 & 8 \\
1 & 1 & 3 & 0 \\
0 & 6 & 3 & 6 \\
0 & 0 & 4 & 2
\end{bmatrix}
\quad \text{det } C = \ldots
\]

\[
\begin{bmatrix}
-1 & 5 & 0 & 8 \\
2 & 2 & 6 & 0 \\
1 & 7 & 6 & 6 \\
1 & 1 & 7 & 2
\end{bmatrix}
\quad \text{det } D = \ldots
\]

\[
\begin{bmatrix}
1 & 1 & 7 & 2 \\
-1 & 5 & 0 & 8 \\
1 & 7 & 6 & 6 \\
2 & 2 & 6 & 0
\end{bmatrix}
\quad \text{det } E = \ldots
\]

Again, \( A \) is upper triangular, with \( \text{det } A = 16 \)

Obtain \( B \) from \( A \) by swapping rows 1 and 4 (change sign) and multiply row 2 by 3 (multiply \( \text{det} \) by 3), so \( \text{det } B = -3 \times 16 = -48 \).

Obtain \( C \) from \( B \) by adding row 2 to row 1 (no change), then swap rows 2 and 3 (-det) and then swap rows 2 and 4 (-det). so \( \text{det } C = \text{det } B = -48 \).

Obtain \( D \) from \( C \) by subtracting row 2 from row 1 (no change), adding row 2 to row 3 (no change), adding row 2 to row 4 (no change) and then doubling row 2 (multiplies \( \text{det} \) by 2), so \( \text{det } D = 2 \times -48 = -96 \).

Obtain \( E \) from \( D \) by swapping rows 1 and 2, then swapping rows 1 and 4, so two sign changes and \( \text{det } E = \text{det } D = -96 \).
each of the others should require little calculation, taking into account the one before.

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
0 & 3 & 2 & 1 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 5 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 5 & 5 & 5 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 5 \\
0 & 6 & 4 & 2 \\
\end{bmatrix}
\]

\[
C = \begin{bmatrix}
1 & -1 & 1 & 3 \\
0 & 0 & -2 & -8 \\
0 & 6 & 4 & 7 \\
0 & 6 & 4 & 2 \\
\end{bmatrix}
\]

\[
D = \begin{bmatrix}
1 & -1 & 1 & 3 \\
-2 & 2 & -4 & -14 \\
1 & 5 & 5 & 10 \\
0 & 3 & 2 & 1 \\
\end{bmatrix}
\]

\[
E = \begin{bmatrix}
1 & 5 & 5 & 10 \\
0 & 3 & 2 & 1 \\
-2 & 2 & -4 & -14 \\
1 & -1 & 1 & 3 \\
\end{bmatrix}
\]

\[
A \text{ is upper triangular, det is product of diagonal elements, so det } A = 15.
\]

Obtain \(B\) from \(A\) by adding row 2 to row 1 (no change in det), swap row 2 and row 3, and then row 3 and row 4 (two sign flips), and then multiply row 4 by 2, so det \(B = 2 \text{ det } A = 30\).

Obtain \(C\) from \(B\) by adding row 4 to row 1 (no change), multiplying row 2 by \(-2\) (multiplies det by \(-2\)), and adding row 4 to row 3 (no change), so det \(C = -2 \text{ det } B = -60\).

Obtain \(D\) from \(C\) by subtracting 2 times row 1 from row 2 (no change), adding row 1 to row 3 (no change) and multiplying row 4 by \(\frac{1}{2}\) (multiplies det by \(\frac{1}{2}\)) so det \(D = \frac{1}{2} \text{ det } C = -30\).

Obtain \(E\) from \(D\) by swapping rows 1 and 3, then swapping rows 2 and 4, and then swapping rows 3 and 4 (so three sign flips), so det \(E = - \text{ det } D = 30\).

1. Compute the determinants of the following matrices. The first one should be easy and each of the others should require little calculation, taking into account the one before.

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 2 & 3 & 4 \\
\end{bmatrix}
\]
\[
B = \begin{bmatrix}
2 & 2 & 3 & 4 \\
-4 & -4 & 0 & 0 \\
0 & -2 & -2 & -4 \\
1 & 2 & 3 & 4
\end{bmatrix}
\quad \text{det } B = \Box
\]
\[
C = \begin{bmatrix}
-2 & -2 & 3 & 4 \\
1 & 2 & 3 & 4 \\
-4 & -4 & 0 & 0 \\
0 & -2 & -2 & -4
\end{bmatrix}
\quad \text{det } C = \Box
\]
\[
D = \begin{bmatrix}
-2 & 0 & 5 & 8 \\
5 & 6 & 3 & 4 \\
1 & 1 & 0 & 0 \\
0 & 2 & 2 & 4
\end{bmatrix}
\quad \text{det } D = \Box
\]
\[
E = \begin{bmatrix}
4 & 0 & -10 & -16 \\
1 & 1 & 0 & 0 \\
0 & 2 & 2 & 4 \\
5 & 6 & 3 & 4
\end{bmatrix}
\quad \text{det } E = \Box
\]

A is lower triangular, det is product of diagonal elements so det \( A = 8 \).

Obtain \( B \) from \( A \) by adding row 4 to row 1 (no change), multiplying row 2 by \(-2\) (multiplies det by \(-2\)), and subtracting row 4 from row 3 (no change), so det \( B = -2 \cdot \text{det } A = -16 \).

Obtain \( C \) from \( B \) by adding row 2 to row 1 (no change), swapping rows 2 and 4 and then swapping rows 3 and 4 (two sign flips), so det \( C = \text{det } B = -16 \).

Obtain \( D \) from \( C \) by subtracting row 4 from row 1 (no change), subtracting row 3 from row 2 (no change), dividing row 3 by \(-4\) (divides the determinant by \(-4\)), and negating row 4 (negates the determinant) so det \( D = -(-\frac{1}{4}) \cdot \text{det } C = -4 \).

Obtain \( E \) from \( D \) by multiplying row 1 by \(-2\) (multiplies determinant by \(-2\)), swapping rows 2 and 3 and then swapping rows 3 and 4 (two sign flips), so det \( E = -2 \cdot \text{det } D = 8 \).

2. Find a basis for the kernel (nullspace) of the matrix

\[
M = \begin{bmatrix}
1 & 2 & 0 & 1 & 3 \\
0 & 0 & 1 & 2 & 1 \\
2 & 4 & 1 & 4 & 7
\end{bmatrix}
\]

What is the rank of \( M \)?

Row reduce:

\[
\begin{bmatrix}
1 & 2 & 0 & 1 & 3 \\
0 & 0 & 1 & 2 & 1 \\
2 & 4 & 1 & 4 & 7
\end{bmatrix} \xrightarrow{R_3 \rightarrow \text{R}_3 - 2 \text{R}_1} \begin{bmatrix}
1 & 2 & 0 & 1 & 3 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 1 & 2 & 1
\end{bmatrix} \xrightarrow{R_3 \rightarrow \text{R}_3 - \text{R}_2} \begin{bmatrix}
1 & 2 & 0 & 1 & 3 \\
0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
so the rank of $M$ is 2 (two nonzero rows in echelon form) and the free variables are $x_2$, $x_4$ and $x_5$. A basis for the kernel is

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$ 

2. Find a basis for the kernel (nullspace) of the matrix

$$M = \begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & -1 & -7 \end{bmatrix}$$

What is the rank of $M$?

Row reduce:

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 1 & 3 & 2 & -1 & -7 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 - R_1} \begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & -1 & -8 \end{bmatrix} \xrightarrow{R_3 \rightarrow R_3 + 8R_2} \begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so the rank of $M$ is 3 (three nonzero rows in echelon form) and the free variables are $x_2$ and $x_3$. A basis for the kernel is

$$\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

2. Find a basis for the kernel (nullspace) of the matrix

$$M = \begin{bmatrix} 1 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

What is the rank of $M$?

2. Find a basis for the kernel (nullspace) of the matrix

$$M = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$
What is the rank of $M$?

3. For each of the following subsets of the vector space $M_{2 \times 2}$ of 2-by-2 matrices, say whether or not it is a (vector) subspace of $M_{2 \times 2}$. If it is not a subspace, explain why not. If it is a subspace, give its dimension and a basis for the subspace.

(a) The set of matrices in $M_{2 \times 2}$ with zero trace (i.e., $A$ such that $\text{tr}(A) = 0$).

Yes, this is a subspace, of dimension 3 and a basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$  

(b) The set of matrices in $M_{2 \times 2}$ with zero determinant (i.e., $A$ such that $\text{det}(A) = 0$).

This is not a subspace, since the two matrices $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are in the set, but their sum, which is the identity matrix, is not.

(c) The set of diagonal matrices in $M_{2 \times 2}$.

This is a subspace of dimension 2 and a basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$  

(c) The set of matrices in $M_{2 \times 2}$ with zeroes on the diagonal.

This is a subspace of dimension 2 and a basis is

$$\left\{ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$  

(d) The set of matrices $A \in M_{2 \times 2}$ such that $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This is a subspace of dimension 2 and a basis is

$$\left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \right\}.$$
(d) The set of matrices $A \in M_{2\times 2}$ such that $A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

This is a subspace of dimension 2 and a basis is

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

4. Consider the matrix $A(k) = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 1 & 0 \\ 1 & 2 & k \end{bmatrix}$.

(a) There is a value of $k$ for which the rank of the matrix $A(k)$ is less than three. What is it?

(b) For this value of $k$, find a basis for the nullspace of $A(k)$.

(a) The determinant of $A(k)$ is $k + 0 + 12 - 0 - 2k - 3 = 9 - k$, so if $k = 9$ then the rank of $A(k)$ is less than 3.

(b) Row reduce:

\[
\begin{bmatrix}
1 & 1 & 3 \\
2 & 1 & 0 \\
1 & 2 & 9
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - 2R_3}
\begin{bmatrix}
1 & 1 & 3 \\
0 & -1 & -6 \\
0 & 1 & 6
\end{bmatrix}
\xrightarrow{R_3 \rightarrow R_3 - R_1}
\begin{bmatrix}
1 & 1 & 3 \\
0 & -1 & -6 \\
0 & 1 & 6
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - R_1}
\begin{bmatrix}
1 & 1 & 3 \\
0 & 0 & 6 \\
0 & 0 & 0
\end{bmatrix}
\]

So a basis for the kernel of $A(9)$ is $\left\{ \begin{bmatrix} -3 \\ -6 \\ 1 \end{bmatrix} \right\}$.

4. Consider the matrix $A(k) = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & k \end{bmatrix}$.

(a) There is a value of $k$ for which the rank of the matrix $A(k)$ is less than three. What is it?

(b) For this value of $k$, find a basis for the nullspace of $A(k)$.

(a) The determinant of $A(k)$ is $k + 0 + 8 - 0 - 2k - 2 = 6 - k$, so if $k = 6$ then the rank of $A(k)$ is less than 3.
(b) Row reduce:

\[
\begin{bmatrix}
1 & 1 & 2 \\
2 & 1 & 0 \\
1 & 2 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 2 \\
0 & -1 & -4 \\
0 & 1 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

So a basis for the kernel of \( A(6) \) is \( \left\{ \begin{bmatrix}
-2 \\
-4 \\
1
\end{bmatrix} \right\} \).

4. Consider the matrix \( A(k) = \begin{bmatrix}
1 & 1 & 4 \\
2 & 1 & 0 \\
1 & 2 & k
\end{bmatrix} \).

(a) There is a value of \( k \) for which the rank of the matrix \( A(k) \) is less than three. What is it?

(b) For this value of \( k \), find a basis for the nullspace of \( A(k) \).

(a) The determinant of \( A(k) \) is \( k + 0 + 16 - 0 - 2k - 4 = 12 - k \), so if \( k = 12 \) then the rank of \( A(k) \) is less than 3.

(b) Row reduce:

\[
\begin{bmatrix}
1 & 1 & 4 \\
2 & 1 & 0 \\
1 & 2 & 12
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 4 \\
0 & -1 & -8 \\
0 & 1 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 4 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

So a basis for the kernel of \( A(12) \) is \( \left\{ \begin{bmatrix}
-4 \\
-8 \\
1
\end{bmatrix} \right\} \).

4. Consider the matrix \( A(k) = \begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & 0 \\
-1 & 2 & k
\end{bmatrix} \).

(a) There is a value of \( k \) for which the rank of the matrix \( A(k) \) is less than three. What is it?

(b) For this value of \( k \), find a basis for the nullspace of \( A(k) \).

(a) The determinant of \( A(k) \) is \( k + 0 + 4 - 0 - 2k + 1 = 5 - k \), so if \( k = 5 \) then the rank of \( A(k) \) is less than 3.
(b) Row reduce:

\[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 1 & 0 \\
-1 & 2 & 5 \\
\end{bmatrix}
\xrightarrow{R_2 \rightarrow R_2 - 2R_1} 
\begin{bmatrix}
1 & 1 & 1 \\
0 & -1 & -2 \\
0 & 3 & 6 \\
\end{bmatrix}
\xrightarrow{R_3 \rightarrow R_3 + R_1} 
\begin{bmatrix}
1 & 1 & 1 \\
0 & -1 & -2 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

So a basis for the kernel of \( A(5) \) is \( \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \).

5. For what value of \( b \) does the following system of equations have a solution? Find the solution for that value of \( b \):

\[
\begin{align*}
3x_1 + 2x_2 &= 7 \\
4x_1 - x_2 &= 2 \\
x_1 + 2x_2 &= b \\
\end{align*}
\]

Just solve the first two equations as two equations in two unknowns (no fancy linear algebra needed): add 2 times the second equation to the first to get \( 11x_1 = 11 \), so \( x_1 = 1 \), and then (from the second equation) \( x_2 = 4 - 2 = 2 \). The third equation will then contradict the first two unless \( b = 1 + 4 = 5 \). So:

\( b = 5, \ x_1 = 1, \ x_2 = 2. \)

5. For what value of \( b \) does the following system of equations have a solution? Find the solution for that value of \( b \):

\[
\begin{align*}
2x_1 + 3x_2 &= 7 \\
4x_1 - x_2 &= 7 \\
x_1 + 3x_2 &= b \\
\end{align*}
\]

Just solve the first two equations as two equations in two unknowns (no fancy linear algebra needed): add 3 times the second equation to the first to get \( 14x_1 = 28 \), so \( x_1 = 2 \), and then (from the second equation) \( x_2 = 8 - 7 = 1 \). The third equation will then contradict the first two unless \( b = 2 + 3 = 5 \). So:

\( b = 5, \ x_1 = 2, \ x_2 = 1. \)

5. For what value of \( b \) does the following system of equations have a solution? Find the solution for that value of \( b \):

\[
\begin{align*}
3x_1 + x_2 &= 9 \\
4x_1 - 2x_2 &= 2 \\
3x_1 + x_2 &= b \\
\end{align*}
\]
Just solve the first two equations as two equations in two unknowns (no fancy linear algebra needed): add 2 times the first equation to the second to get $10x_1 = 20$, so $x_1 = 2$, and then (from the first equation) $x_2 = 9 - 6 = 3$. The third equation will then contradict the first two unless $b = 6 + 3 = 9$. So:

$$b = 9, \ x_1 = 2, \ x_2 = 3.$$ 

5. For what value of $b$ does the following system of equations have a solution? Find the solution for that value of $b$:

$$\begin{align*}
2x_1 + x_2 &= 8 \\
3x_1 - x_2 &= 7 \\
x_1 - 2x_2 &= b
\end{align*}$$

Just solve the first two equations as two equations in two unknowns (no fancy linear algebra needed): add the second equation to the first to get $5x_1 = 15$, so $x_1 = 3$, and then (from the second equation) $x_2 = 9 - 7 = 2$. The third equation will then contradict the first two unless $b = 3 - 4 = -1$. So:

$$b = -1, \ x_1 = 3, \ x_2 = 2.$$ 

6. Let $S = \{1, x, x^2, x^3\}$ be the standard basis for the vector space $P_3$ of polynomials of degree less than or equal to 3.

$B = \{1 + x + x^2, 1 + x + x^3, 1 + x^2 + x^3, x + x^2 + x^3\}$ is another basis for $P_3$. You may take this for granted (your solution to part (b) below will imply that $B$ is a basis).

(a) What is the change-of-basis matrix $P_{S \to B}$ (in other words how do you go from expressing a polynomial as $a_1(1 + x + x^2) + a_2(1 + x + x^3) + a_3(1 + x^2 + x^3) + a_4(x + x^2 + x^3)$ to expressing it as $b_1(1) + b_2(x) + b_3(x^2) + b_4(x^3)$)?

(b) What is the change-of-basis matrix $P_{B \to S}$?

(c) What is the matrix the represents the linear mapping that sends $p(x)$ to $p'(x) + 3p(x)$ with respect to the standard basis $S$?

(d) (Extra Credit) What is the matrix that represents the linear mapping in part (c) with respect to the basis $B$? (If you choose to do this one, show your work on the back of the page.)

(a) Columns of $P_{S \to B}$ are the basis vectors, so

$$P_{S \to B} = \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}.$$
(b) We have $P_{B^*S} = P_{S^*B}^{-1}$ so row reduce:

$$
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Therefore

$$
P_{B^*S} = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\
\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}
$$

(c) Suppose $p(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$, so the vector representing $p$ with respect to the standard basis is $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$. Then $p' + 3p = (3a_1 + a_2) + (3a_2 + 2a_3)x + (3a_3 + 3a_4)x^2 + 3a_4 x^3$, which is represented by the vector $\begin{pmatrix} 3a_1 + a_2 \\ 3a_2 + 2a_3 \\ 3a_3 + 3a_4 \\ 3a_4 \end{pmatrix}$ in the standard basis. The matrix that sends the first vector to the second is

$$
M_S = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}
$$
(d) To get the matrix of the transformation with respect to the basis $B$ we calculate:

$$M_B = P_{B\to S}M_SP_{S\to B} = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\
\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}\begin{bmatrix}
3 & 1 & 0 & 0 \\
0 & 3 & 2 & 0 \\
0 & 0 & 3 & 3 \\
0 & 0 & 0 & 3
\end{bmatrix}\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\
\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}\begin{bmatrix}
4 & 4 & 3 & 1 \\
5 & 3 & 2 & 5 \\
3 & 3 & 6 & 6 \\
0 & 3 & 3 & 3
\end{bmatrix} = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & -\frac{4}{3} & -1 \\
-\frac{1}{3} & \frac{8}{3} & 0 \\
0 & \frac{1}{3} & \frac{5}{3} & 4
\end{bmatrix}\begin{bmatrix}
4 & 4 & \frac{5}{3} & 2 \\
1 & 4 & \frac{4}{3} & -\frac{1}{3}
\end{bmatrix}$$

6. Let $S = \{1, x, x^2, x^3\}$ be the standard basis for the vector space $P_3$ of polynomials of degree less than or equal to 3.

$B = \{1 + x + x^2, 1 + x + x^3, 1 + x^2 + x^3, x + x^2 + x^3\}$ is another basis for $P_3$. You may take this for granted (your solution to part (b) below will imply that $B$ is a basis).

(a) What is the change-of-basis matrix $P_{S\to B}$ (in other words how do you go from expressing a polynomial as $a_1(1 + x + x^2) + a_2(1 + x + x^3) + a_3(1 + x^2 + x^3) + a_4(x + x^2 + x^3)$ to expressing it as $b_1(1) + b_2(x) + b_3(x^2) + b_4(x^3)$)?

(b) What is the change-of-basis matrix $P_{B\to S}$?

(c) What is the matrix the represents the linear mapping that sends $p(x)$ to $2p'(x) + p(x)$ with respect to the standard basis $S$?

(d) (Extra Credit) What is the matrix that represents the linear mapping in part (c) with respect to the basis $B$? (If you choose to do this one, show your work on the back of the page.)

Parts (a) and (b) are the same as above.

(c) Suppose $p(x) = a_1 + a_2x + a_3x^2 + a_4x^3$, so the vector representing $p$ with respect to the standard basis is $\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{bmatrix}$. Then $2p' + p = (a_1 + 2a_2) + (a_2 + 4a_3)x + (a_3 + 6a_4)x^2 + a_4x^3$, which is represented by the vector $\begin{bmatrix}
a_1 + 2a_2 \\
a_2 + 4a_3 \\
a_3 + 6a_4 \\
a_4
\end{bmatrix}$ in the standard basis. The matrix that sends the
first vector to the second is
\[ M_S = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

(d) To get the matrix of the transformation with respect to the basis \( B \) we calculate:
\[
M_B = P_{B\to S} M_S P_{S\to B} = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\
\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 1 & 4 & 0 \\
0 & 0 & 1 & 6 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\
\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\
\frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
3 & 3 & 1 & 2 \\
5 & 1 & 4 & 5 \\
1 & 6 & 7 & 7 \\
0 & 1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
3 & \frac{10}{3} & \frac{4}{3} \\
2 & -\frac{8}{3} & -\frac{8}{3} & -2 \\
-2 & \frac{3}{3} & \frac{1}{3} & 0 \\
0 & \frac{3}{3} & \frac{10}{3} & 3
\end{bmatrix}.
\]

6. Let \( S = \{1, x, x^2, x^3\} \) be the standard basis for the vector space \( P_3 \) of polynomials of degree less than or equal to 3.

\( B = \{1 + x + x^2, 1 + x + 3, 1 + x^2 + 3, x + x^2 + 3\} \) is another basis for \( P_3 \). You may take this for granted (your solution to part (b) below will imply that \( B \) is a basis).

(a) What is the change-of-basis matrix \( P_{S\to B} \) (in other words how do you go from expressing a polynomial as \( a_1(1 + x + x^2) + a_2(1 + x + 3) + a_3(1 + x^2 + 3) + a_4(x + x^2 + 3) \) to expressing it as \( b_1(1) + b_2(x) + b_3(x^2) + b_4(x^3) \)?

(b) What is the change-of-basis matrix \( P_{B\to S} \)?

(c) What is the matrix the represents the linear mapping that sends \( p(x) \) to \( p''(x) + p(x) \) with respect to the standard basis \( S \)?

(d) \( \text{(Extra Credit)} \) What is the matrix that represents the linear mapping in part (c) with respect to the basis \( B \)? (If you choose to do this one, show your work on the back of the page.)

Parts (a) and (b) are the same as above.

(c) Suppose \( p(x) = a_1 + a_2x + a_3x^2 + a_4x^3 \), so the vector representing \( p \) with respect to the standard basis is \( \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \). Then \( p'' + p = (a_1 + 2a_3) + (a_2 + 6a_4)x + a_3x^2 + a_4x^3 \), which
is represented by the vector \[
\begin{bmatrix}
  a_1 + 2a_3 \\
  a_2 + 6a_4 \\
  a_3 \\
  a_4
\end{bmatrix}
\] in the standard basis. The matrix that sends the first vector to the second is
\[
M_S = \begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 1 & 0 & 6 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix}.
\]

(d) To get the matrix of the transformation with respect to the basis \(B\) we calculate:
\[
M_B = P_{B\leftrightarrow S}M_SP_{S\leftrightarrow B} = \begin{bmatrix}
  \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\
  \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\
  \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\
  -\frac{2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{bmatrix} \begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 1 & 0 & 6 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{bmatrix} = \begin{bmatrix}
  \frac{5}{3} & 2 & \frac{8}{3} & \frac{8}{3} \\
  \frac{2}{3} & 3 & \frac{8}{3} & \frac{8}{3} \\
  \frac{2}{3} & -4 & \frac{7}{3} & -\frac{10}{3} \\
  -\frac{4}{3} & 2 & \frac{2}{3} & \frac{5}{3}
\end{bmatrix}
\]

6. Let \(S = \{1, x, x^2, x^3\}\) be the standard basis for the vector space \(P_3\) of polynomials of degree less than or equal to 3.

\(B = \{1 + x + x^2, 1 + x + x^3, 1 + x^2 + x^3, x + x^2 + x^3\}\) is another basis for \(P_3\). You may take this for granted (your solution to part (b) below will imply that \(B\) is a basis).

(a) What is the change-of-basis matrix \(P_{S\leftrightarrow B}\) (in other words how do you go from expressing a polynomial as \(a_1(1 + x + x^2) + a_2(1 + x + x^3) + a_3(1 + x^2 + x^3) + a_4(x + x^2 + x^3)\) to expressing it as \(b_1(1) + b_2(x) + b_3(x^2) + b_4(x^3)\)?)

(b) What is the change-of-basis matrix \(P_{B\leftrightarrow S}\)?

(c) What is the matrix the represents the linear mapping that sends \(p(x)\) to \(p''(x) + p'(x)\) with respect to the standard basis \(S\)?

(d) (Extra Credit) What is the matrix that represents the linear mapping in part (c) with respect to the basis \(B\)? (If you choose to do this one, show your work on the back of the page.)

Parts (a) and (b) are the same as above.

(c) Suppose \(p(x) = a_1 + a_2x + a_3x^2 + a_4x^3\), so the vector representing \(p\) with respect to
the standard basis is

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4 \\
\end{bmatrix}
\]

. Then \( p'' + p' = (a_2 + 2a_3) + (2a_3 + 6a_4)x + 3a_4x^2 \), which is represented by the vector

\[
\begin{bmatrix}
a_2 + 2a_3 \\
2a_3 + 6a_4 \\
3a_4 \\
0 \\
\end{bmatrix}
\]

in the standard basis. The matrix that sends the first vector to the second is

\[
M_S = \begin{bmatrix}
0 & 1 & 2 & 0 \\
0 & 0 & 2 & 6 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(d) To get the matrix of the transformation with respect to the basis \( B \) we calculate:

\[
M_B = P_{B\leftarrow S}M_SP_{S\leftarrow B} = \begin{bmatrix}
1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{bmatrix}\begin{bmatrix}
0 & 1 & 2 & 0 \\
0 & 0 & 2 & 6 \\
0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-2}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{1}{3} & \frac{1}{3} & \frac{-2}{3} & \frac{1}{3} & \frac{1}{3} \\
\frac{-2}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
\end{bmatrix}\begin{bmatrix}
3 & 1 & 2 & 3 \\
2 & 6 & 8 & 8 \\
0 & 3 & 3 & 3 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} = \begin{bmatrix}
\frac{5}{3} & \frac{10}{3} & \frac{13}{3} & \frac{14}{3} \\
\frac{5}{3} & \frac{1}{3} & \frac{4}{3} & \frac{5}{3} \\
\frac{1}{3} & \frac{-8}{3} & \frac{-11}{3} & \frac{-10}{3} \\
\frac{-4}{3} & \frac{7}{3} & \frac{5}{3} & \frac{3}{3} \\
\end{bmatrix}
\]
1. (a) Find the general solution to the differential equation

\[ y'' + 6y' + 10y = 50t + 10. \]

(b) Find the unique solution of the differential equation in part (a) that also satisfies the initial conditions

\[ y(0) = 0, \quad y'(0) = 0. \]

(a) First, to find the solution of the homogeneous equation we need the roots of \( r^2 + 6r + 10 \), which are

\[ r = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i. \]

So the solution of the homogeneous equation is

\[ y_c = c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t. \]

Next, we use undetermined coefficients to find a particular solution and guess that \( y_p = At + B \). Then \( y_p' = A \) and \( y_p'' = 0 \) so

\[ y''_p + 6y'_p + 10y_p = 10At + (10B + 6A) \]

Since this is supposed to equal \( 50t + 10 \), we have that \( A = 5 \) and \( B = -2 \). So the general solution is

\[ y = 5t - 2 + c_1 e^{-3t} \cos t + c_2 e^{-3t} \sin t. \]

(b) First of all, \( y' = 5 - 3c_1 e^{-3t} \cos t - c_2 e^{-3t} \sin t - 3c_2 e^{-3t} \sin t + c_2 e^{-3t} \cos t \). So \( y(0) = -2 + c_1 \) and \( y'(0) = 5 - 3c_1 + c_2 \). For both to be zero we need \( c_1 = 2 \) and \( c_2 = 1 \). So the solution is

\[ y(t) = 5t - 2 + 2e^{-3t} \cos t + e^{-3t} \sin t. \]
So the solution of the homogeneous equation is \( y_c = c_1 e^{-4t} \cos 3t + c_2 e^{-4t} \sin 3t \). Next, we use undetermined coefficients to find a particular solution and guess that \( y_p = At + B \). Then \( y''_p = A \) and \( y'''_p = 0 \) so\[ y''_p + 8y'_p + 25y_p = 25At + (25B + 8A) \]
Since this is supposed to equal \( 125t + 15 \), we have that \( A = 5 \) and \( B = -1 \). So the general solution is\[ y = 5t - 1 + c_1 e^{-4t} \cos 3t + c_2 e^{-4t} \sin 3t. \]

(b) First of all, \( y' = 5 - 4c_1 e^{-4t} \cos 3t - 3c_1 e^{-4t} \sin 3t - 4c_2 e^{-4t} \sin 3t + 3c_2 e^{-4t} \cos 3t \). So \( y(0) = -1 + c_1 \) and \( y'(0) = 5 - 4c_1 + 3c_2 \). For both to be zero we need \( c_1 = 1 \) and \( c_2 = -\frac{1}{3} \). So the solution is\[ y(t) = 5t - 1 + e^{-4t} \cos 3t - \frac{1}{3} e^{-4t} \sin 3t. \]

1. (a) Find the general solution to the differential equation\[ y'' + 4y' + 5y = 25t + 5. \]

(b) Find the unique solution of the differential equation in part (a) that also satisfies the initial conditions\[ y(0) = 0, \quad y'(0) = 0. \]

(a) First, to find the solution of the homogeneous equation we need the roots of \( r^2 + 4r + 5 \), which are\[ r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i. \]
So the solution of the homogeneous equation is \( y_c = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t \). Next, we use undetermined coefficients to find a particular solution and guess that \( y_p = At + B \). Then \( y'_p = A \) and \( y''_p = 0 \) so\[ y''_p + 4y'_p + 5y_p = 5At + (5B + 4A) \]
Since this is supposed to equal \( 25t + 5 \), we have that \( A = 5 \) and \( B = -3 \). So the general solution is\[ y = 5t - 3 + c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t. \]

(b) First of all, \( y' = 5 - 2c_1 e^{-2t} \cos t - c_1 e^{-2t} \sin t - 2c_2 e^{-2t} \sin t + c_2 e^{-2t} \cos t \). So \( y(0) = -3 + c_1 \) and \( y'(0) = 5 - 2c_1 + c_2 \). For both to be zero we need \( c_1 = 3 \) and \( c_2 = 1 \). So the solution is\[ y(t) = 5t - 3 + 3e^{-2t} \cos t + e^{-2t} \sin t. \]

1. (a) Find the general solution to the differential equation\[ y'' + 10y' + 29y = 29t + 39. \]
(b) Find the unique solution of the differential equation in part (a) that also satisfies the initial conditions

\[ y(0) = 0, \quad y'(0) = 0. \]

(a) First, to find the solution of the homogeneous equation we need the roots of \( r^2 + 10r + 29 \), which are

\[ r = \frac{-10 \pm \sqrt{100 - 116}}{2} = -5 \pm 2i. \]

So the solution of the homogeneous equation is

\[ y_c = c_1 e^{-5t} \cos 2t + c_2 e^{-5t} \sin 2t. \]

Next, we use undetermined coefficients to find a particular solution and guess that \( y_p = At + B \). Then \( y'_p = A \) and \( y''_p = 0 \) so

\[ y_p'' + 10y'_p + 29y_p = 29At + (29B + 10A) \]

Since this is supposed to equal \( 29t + 39 \), we have that \( A = 1 \) and \( B = 1 \). So the general solution is

\[ y = t + 1 + c_1 e^{-5t} \cos 2t + c_2 e^{-5t} \sin 2t. \]

(b) First of all, \( y' = 1 - 5c_1 e^{-5t} \cos 2t - 2c_1 e^{-5t} \sin 2t - 5c_2 e^{-5t} \sin 2t + 2c_2 e^{-5t} \cos 2t \). So \( y(0) = 1 + c_1 \) and \( y'(0) = 1 - 5c_1 + 2c_2 \). For both to be zero we need \( c_1 = -1 \) and \( c_2 = -3 \). So the solution is

\[ y(t) = t + 1 - e^{-5t} \cos 2t - 3e^{-5t} \sin 2t. \]
(b) First of all, \( y' = 2 - 6c_1e^{-6t} \cos 2t - 2c_1e^{-6t} \sin 2t - 6c_2e^{-6t} \sin 2t + 2c_2e^{-6t} \cos 2t \). So \( y(0) = c_1 \) and \( y'(0) = 2 - 6c_1 + 2c_2 \). For both to be zero we need \( c_1 = 0 \) and \( c_2 = -1 \). So the solution is

\[
y(t) = 2t - e^{-6t} \sin 2t.
\]

2. Consider the differential equation

\[
y'' + ky' + 16y = 0.
\]

(a) For which values of \( k \) will all of the solutions of the equation have infinitely many values of \( t \) for which the solution \( y(t) \) satisfies \( y(t) = 0? \)

(b) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to +\infty? \)

(c) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to -\infty? \)

We can use the roots of \( r^2 + kr + 16 = 0 \), namely

\[
r = \frac{-k \pm \sqrt{k^2 - 64}}{2}
\]

to analyze the behavior of the solutions.

(a) The solutions will oscillate (that’s what this means) if there are sine and cosine factors, so the roots would have to be complex. This would happen if \( k^2 - 64 < 0 \), i.e., if \( -8 < k < 8 \).

(b) This will happen if the coefficient of \( t \) in the exponential factors is not positive (so the solutions do not become infinite as \( t \) gets large). If the solutions are complex, then the coefficient is \(-k/2\) so this is not positive for \( 0 \leq k < 8 \). If \( k \geq 8 \) then both roots will be real and negative and the solutions will decay as well. So the solutions remain bounded as \( t \to \infty \) if \( k \geq 0 \).

(c) This will happen if the coefficient of \( t \) in the exponential factors is not negative (so the solutions do not become infinite as \( t \) becomes more and more negative). If the solutions are complex, the coefficient is \(-k/2\), and this is not negative for \(-8 < k \leq 0 \). If \( k \leq -8 \) then both roots will be real and positive and the solutions will decay as \( t \to -\infty \). So the solutions remain bounded as \( t \to -\infty \) if \( k \leq 0 \).
(a) For which values of \( k \) will all of the solutions of the equation have infinitely many values of \( t \) for which the solution \( y(t) \) satisfies \( y(t) = 0 \)?

(b) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to +\infty \)?

(c) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to -\infty \)?

We can use the roots of \( r^2 + 6r + k = 0 \), namely

\[
r = \frac{-6 \pm \sqrt{36 - 4k}}{2}
\]

to analyze the behavior of the solutions.

(a) The solutions will oscillate (that’s what this means) if there are sine and cosine factors, so the roots would have to be complex. This would happen if \( 36 - 4k < 0 \), i.e., if \( k > 9 \).

(b) This will happen if the coefficient of \( t \) in the exponential factors is not positive (so the solutions do not become infinite as \( t \) gets large). If the solutions are complex, then the coefficient is \(-3\) and this is not positive. So \( k > 9 \) works. If \( k \leq 9 \) then both roots will be real and negative and the solutions will decay provided \(-6 + \sqrt{36 - 4k} < 0\). In other words, we need \( \sqrt{36 - 4k} < 6 \), which means \( k \) should be positive (and less than or equal to 9). The borderline case \( k = 0 \) works as well, since then there are constant solutions which of course are bounded. So the solutions remain bounded as \( t \to \infty \) if \( k \geq 0 \).

(c) This will happen if the coefficient of \( t \) in the exponential factors is not negative (so the solutions do not become infinite as \( t \) becomes more and more negative). But the roots are \(-3 \pm \text{something}\), so one or both of the roots is always negative or has negative real part. So there is no value of \( k \) for which all the solutions remain bounded as \( t \to -\infty \).

2. Consider the differential equation

\[
y'' + ky' + 9y = 0.
\]

(a) For which values of \( k \) will all of the solutions of the equation have infinitely many values of \( t \) for which the solution \( y(t) \) satisfies \( y(t) = 0 \)?

(b) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to +\infty \)?

(c) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to -\infty \)?

We can use the roots of \( r^2 + kr + 9 = 0 \), namely

\[
r = \frac{-k \pm \sqrt{k^2 - 36}}{2}
\]
(a) The solutions will oscillate (that’s what this means) if there are sine and cosine factors, so the roots would have to be complex. This would happen if \( k^2 - 36 < 0 \), i.e., if \(-6 < k < 6\).

(b) This will happen if the coefficient of \( t \) in the exponential factors is not positive (so the solutions do not become infinite as \( t \) gets large). If the solutions are complex, then the coefficient is \(-k/2\) so this is not positive for \( 0 \leq k < 6 \). If \( k \geq 6 \) then both roots will be real and negative and the solutions will decay as well. So the solutions remain bounded as \( t \to \infty \) if \( k \geq 0 \).

(c) This will happen if the coefficient of \( t \) in the exponential factors is not negative (so the solutions do not become infinite as \( t \) becomes more and more negative). If the solutions are complex, the coefficient is \(-k/2\), and this is not negative for \(-6 < k \leq 0 \). If \( k \leq -6 \) then both roots will be real and positive and the solutions will decay as \( t \to -\infty \). So the solutions remain bounded as \( t \to -\infty \) if \( k \leq 0 \).

---

2. Consider the differential equation

\[ y'' + 2y' + ky = 0. \]

(a) For which values of \( k \) will all of the solutions of the equation have infinitely many values of \( t \) for which the solution \( y(t) \) satisfies \( y(t) = 0 \)?

(b) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to +\infty \)?

(c) For which values of \( k \) will all of the solutions of the equation remain bounded as \( t \to -\infty \)?

We can use the roots of \( r^2 + 2r + k = 0 \), namely

\[ r = \frac{-2 \pm \sqrt{4 - 4k}}{2} \]

to analyze the behavior of the solutions.

(a) The solutions will oscillate (that’s what this means) if there are sine and cosine factors, so the roots would have to be complex. This would happen if \( 4 - 4k < 0 \), i.e., if \( k > 1 \).

(b) This will happen if the coefficient of \( t \) in the exponential factors is not positive (so the solutions do not become infinite as \( t \) gets large). If the solutions are complex, then the coefficient is \(-1\) and this is not positive. So \( k > 1 \) works. If \( k \leq 1 \) then both roots will be real and negative and the solutions will decay provided \(-1 + \sqrt{4 - 4k} < 0 \). In other words, we need \( \sqrt{4 - 4k} < 4 \), which means \( k \) should be positive (and less than or equal to 1). The borderline case \( k = 0 \) works as well, since then there are constant solutions which of course are bounded. So the solutions remain bounded as \( t \to \infty \) if \( k \geq 0 \).
(c) This will happen if the coefficient of $t$ in the exponential factors is not negative (so the solutions do not become infinite as $t$ becomes more and more negative). But the roots are $-1\pm$ something, so one or both of the roots is always negative or has negative real part. So there is no value of $k$ for which all the solutions remain bounded as $t \to -\infty$.

2. Consider the differential equation

$$y'' + 10y' + ky = 0.$$  

(a) For which values of $k$ will all of the solutions of the equation have infinitely many values of $t$ for which the solution $y(t)$ satisfies $y(t) = 0$?

(b) For which values of $k$ will all of the solutions of the equation remain bounded as $t \to +\infty$?

(c) For which values of $k$ will all of the solutions of the equation remain bounded as $t \to -\infty$?

We can use the roots of $r^2 + 10r + k = 0$, namely

$$r = \frac{-10 \pm \sqrt{100 - 4k}}{2}$$

to analyze the behavior of the solutions.

(a) The solutions will oscillate (that’s what this means) if there are sine and cosine factors, so the roots would have to be complex. This would happen if $100 - 4k < 0$, i.e., if $k > 25$.

(b) This will happen if the coefficient of $t$ in the exponential factors is not positive (so the solutions do not become infinite as $t$ gets large). If the solutions are complex, then the coefficient is $-5$ and this is not positive. So $k > 25$ works. If $k \leq 25$ then both roots will be real and negative and the solutions will decay provided $-10 + \sqrt{100 - 4k} < 0$. In other words, we need $\sqrt{100 - 4k} < 10$, which means $k$ should be positive (and less than or equal to 25). The borderline case $k = 0$ works as well, since then there are constant solutions which of course are bounded. So the solutions remain bounded as $t \to \infty$ if $k \geq 0$.

(c) This will happen if the coefficient of $t$ in the exponential factors is not negative (so the solutions do not become infinite as $t$ becomes more and more negative). But the roots are $-5\pm$ something, so one or both of the roots is always negative or has negative real part. So there is no value of $k$ for which all the solutions remain bounded as $t \to -\infty$.

3. Consider the linear homogeneous differential equation

$$ty'' - (2t + 4)y' + (t + 4)y = 0.$$
Since the coefficients of \( y'' \), \( y' \) and \( y \) add up to zero, one solution of this equation is \( e^t \). Find another linearly independent solution.

Use reduction of order, assume that \( y = ve^t \) where \( v \) is an unknown function. So \( y' = ve^t + v'e^t \) and \( y'' + ve^t + 2v'e^t + v''e^t \). We know that all the terms with an undifferentiated \( v \) will cancel when we substitute \( y = ve^t \) into the differential equation, so we won’t write them at all and so:

\[
ty'' - (2t + 4)y' + (t + 4)y = t(2v' e^t + v'' e^t) - (2t + 4)(v'e^t) = e^t(tv'' - 4v')
\]

We need this to be zero, which is a first-order equation for \( w = v' \), namely \( tw' - 4w = 0 \). This equation is separable and we integrate

\[
\int \frac{dw}{w} = 4 \int \frac{dt}{t}
\]

to get \( w = c_1 t^4 \). Integrating this we get \( v = \frac{1}{3} c_1 t^5 + c_2 \). So the general solution of the equation (after absorbing the \( \frac{1}{3} \) into the constant \( c_1 \)) is

\[
y = c_1 t^5 e^t + c_2 e^t.
\]

(In other words, the second linearly independent solution is \( t^5 e^t \).)

3. Consider the linear homogeneous differential equation

\[
ty'' - (2t + 5)y' + (t + 5)y = 0.
\]

Since the coefficients of \( y'' \), \( y' \) and \( y \) add up to zero, one solution of this equation is \( e^t \). Find another linearly independent solution.

Use reduction of order, assume that \( y = ve^t \) where \( v \) is an unknown function. So \( y' = ve^t + v'e^t \) and \( y'' + ve^t + 2v'e^t + v''e^t \). We know that all the terms with an undifferentiated \( v \) will cancel when we substitute \( y = ve^t \) into the differential equation, so we won’t write them at all and so:

\[
ty'' - (2t + 5)y' + (t + 5)y = t(2v' e^t + v'' e^t) - (2t + 5)(v'e^t) = e^t(tv'' - 5v')
\]

We need this to be zero, which is a first-order equation for \( w = v' \), namely \( tw' - 5w = 0 \). This equation is separable and we integrate

\[
\int \frac{dw}{w} = 5 \int \frac{dt}{t}
\]

to get \( w = c_1 t^5 \). Integrating this we get \( v = \frac{1}{6} c_1 t^6 + c_2 \). So the general solution of the equation (after absorbing the \( \frac{1}{6} \) into the constant \( c_1 \)) is

\[
y = c_1 t^6 e^t + c_2 e^t.
\]
(In other words, the second linearly independent solution is $t^6 e^t$.)

3. Consider the linear homogeneous differential equation

$$ty'' - (2t - 4)y' + (t - 4)y = 0.$$ 

Since the coefficients of $y''$, $y'$ and $y$ add up to zero, one solution of this equation is $e^t$. Find another linearly independent solution.

Use reduction of order, assume that $y = ve^t$ where $v$ is an unknown function. So $y' = ve^t + v'e^t$ and $y'' + ve^t + 2v'e^t + v''e^t$. We know that all the terms with an undifferentiated $v$ will cancel when we substitute $y = ve^t$ into the differential equation, so we won’t write them at all and so:

$$ty'' - (2t - 4)y' + (t - 4)y = t(2v'e^t + v''e^t) - (2t - 4)(v'e^t) = e^t(tv'' + 4v')$$

We need this to be zero, which is a first-order equation for $w = v'$, namely $tw' + 4w = 0$. This equation is separable and we integrate

$$\int \frac{dw}{w} = -4 \int \frac{dt}{t}$$

to get $w = c_1 t^{-4}$. Integrating this we get $v = -\frac{1}{3} c_1 t^{-3} + c_2$. So the general solution of the equation (after absorbing the $-\frac{1}{3}$ into the constant $c_1$) is

$$y = c_1 t^{-3} e^t + c_2 e^t.$$

(In other words, the second linearly independent solution is $t^{-3} e^t$.)

3. Consider the linear homogeneous differential equation

$$ty'' - (2t - 3)y' + (t - 3)y = 0.$$ 

Since the coefficients of $y''$, $y'$ and $y$ add up to zero, one solution of this equation is $e^t$. Find another linearly independent solution.

Use reduction of order, assume that $y = ve^t$ where $v$ is an unknown function. So $y' = ve^t + v'e^t$ and $y'' + ve^t + 2v'e^t + v''e^t$. We know that all the terms with an undifferentiated $v$ will cancel when we substitute $y = ve^t$ into the differential equation, so we won’t write them at all and so:

$$ty'' - (2t - 3)y' + (t - 3)y = t(2v'e^t + v''e^t) - (2t - 3)(v'e^t) = e^t(tv'' + 3v')$$

We need this to be zero, which is a first-order equation for $w = v'$, namely $tw' + 3w = 0$. This equation is separable and we integrate

$$\int \frac{dw}{w} = -3 \int \frac{dt}{t}$$
to get $w = c_1 t^{-3}$. Integrating this we get $v = -\frac{1}{2}c_1 t^{-2} + c_2$. So the general solution of the equation (after absorbing the $-\frac{1}{2}$ into the constant $c_1$) is

$$y = c_1 t^{-2} e^t + c_2 e^t.$$ 

(In other words, the second linearly independent solution is $t^{-2} e^t$.)

3. Consider the linear homogeneous differential equation

$$ty'' - (2t + 6)y' + (t + 6)y = 0.$$ 

Since the coefficients of $y''$, $y'$ and $y$ add up to zero, one solution of this equation is $e^t$. Find another linearly independent solution.

Use reduction of order, assume that $y = ve^t$ where $v$ is an unknown function. So $y' = ve^t + v'e^t$ and $y'' = ve^t + 2v'e^t + v''e^t$. We know that all the terms with an undifferentiated $v$ will cancel when we substitute $y = ve^t$ into the differential equation, so we won’t write them at all and so:

$$ty'' - (2t + 6)y' + (t + 6)y = t(2v'e^t + v''e^t) - (2t + 6)(v'e^t) = e^t(tv'' - 6v')$$

We need this to be zero, which is a first-order equation for $w = v'$, namely $tw' - 6w = 0$. This equation is separable and we integrate

$$\int \frac{dw}{w} = 6 \int \frac{dt}{t}$$

to get $w = c_1 t^6$. Integrating this we get $v = \frac{1}{7}c_1 t^7 + c_2$. So the general solution of the equation (after absorbing the $\frac{1}{7}$ into the constant $c_1$) is

$$y = c_1 t^7 e^t + c_2 e^t.$$ 

(In other words, the second linearly independent solution is $t^7 e^t$.)

4. Let $M_n$ be the $n$-by-$n$ matrix all of whose entries are 2. So for example:

$$M_4 = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}.$$ 

(If you are uncomfortable dealing with $M_n$ for arbitrary $n$, you can do the $n = 4$ case and receive most of the credit for this problem.)

(a) What is $\text{tr}(M_n)$?
(b) What is $\det(M_n)$?

(c) What are the eigenvalues and eigenvectors of $M_n$?

(d) Find a diagonal matrix $D$ and an invertible matrix $P$ so that $M_n = PDP^{-1}$.

(e) Extra credit: What is $e^{tM_n}$?

(a) $\text{tr}(M_n) = 2n$

(b) $\det(M_n) = 0$ since there are many identical rows.

(c) Since $M_n$ has rank 1, $\lambda = 0$ is an eigenvalue with multiplicity $n - 1$. And since the trace of $M_n$ is $2n$ (which is the sum of the eigenvalues), the last eigenvalue must be $2n$. The eigenvectors corresponding to $\lambda = 0$ are the ones that sum to zero, so a basis is

$$
\begin{bmatrix}
1 \\
-1 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
, \begin{bmatrix}
1 \\
0 \\
-1 \\
0 \\
\vdots \\
0
\end{bmatrix}
, \begin{bmatrix}
1 \\
0 \\
-1 \\
0 \\
\vdots \\
-1
\end{bmatrix}
, \ldots
, \begin{bmatrix}
1 \\
0 \\
0 \\
0 \\
\vdots \\
-1
\end{bmatrix}
$$

and the eigenvector corresponding to $\lambda = 2n$ is

$$
\begin{bmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
$$

(d) From part (c) we see that

$$
D = \begin{bmatrix}
2n & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\quad \text{and} \quad
P = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & -1 & 0 & 0 & \cdots & 0 \\
1 & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -1
\end{bmatrix}
$$

(e) Let $U_n$ be the $n \times n$ matrix with all 1’s in it, so $M_n = 2U_n$. The $U_n^2 = nU_n$, $U_n^3 = n^2U_n$,
etc., so that $U_n^k = n^{k-1}U_n$. Therefore $M_n^k = (2U_n)^k = 2^kn^{k-1}U_n$ Therefore

$$e^{tM_n} = I + tM_n + \frac{t^2M_n^2}{2!} + \frac{t^3M_n^3}{3!} + \cdots + \frac{t^kM_n^k}{k!} + \cdots$$

$$= I + 2tU_n + \frac{2^2t^2nU_n}{2!} + \frac{2^3t^3n^2U_n}{3!} + \cdots + \frac{2^ktn^kU_n}{k!} + \cdots$$

$$= I + \frac{1}{n}U_n \left( 2tn + \frac{(2tn)^2}{2!} + \frac{(2tn)^3}{3!} + \cdots + \frac{(2tn)^k}{k!} + \cdots \right)$$

$$= I + \frac{e^{2tn} - 1}{n}U_n$$

$$\begin{bmatrix}
1 + \frac{1}{n}(e^{2tn} - 1) & \frac{1}{n}(e^{2tn} - 1) & \frac{1}{n}(e^{2tn} - 1) & \cdots & \frac{1}{n}(e^{2tn} - 1) \\
\frac{1}{n}(e^{2tn} - 1) & 1 + \frac{1}{n}(e^{2tn} - 1) & \frac{1}{n}(e^{2tn} - 1) & \cdots & \frac{1}{n}(e^{2tn} - 1) \\
\frac{1}{n}(e^{2tn} - 1) & \frac{1}{n}(e^{2tn} - 1) & 1 + \frac{1}{n}(e^{2tn} - 1) & \cdots & \frac{1}{n}(e^{2tn} - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n}(e^{2tn} - 1) & \frac{1}{n}(e^{2tn} - 1) & \frac{1}{n}(e^{2tn} - 1) & \cdots & 1 + \frac{1}{n}(e^{2tn} - 1)
\end{bmatrix}$$

4. Let $M_n$ be the $n$-by-$n$ matrix all of whose entries are 3. So for example:

$$M_4 = \begin{bmatrix}
3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3
\end{bmatrix}$$

(If you are uncomfortable dealing with $M_n$ for arbitrary $n$, you can do the $n = 4$ case and receive most of the credit for this problem.)

(a) What is $\text{tr}(M_n)$?

(b) What is $\det(M_n)$?

(c) What are the eigenvalues and eigenvectors of $M_n$?

(d) Find a diagonal matrix $D$ and an invertible matrix $P$ so that $M_n = PD P^{-1}$.

(e) Extra credit: What is $e^{tM_n}$?

(a) $\text{tr}(M_n) = 3n$

(b) $\det(M_n) = 0$ since there are many identical rows.

(c) Since $M_n$ has rank 1, $\lambda = 0$ is an eigenvalue with multiplicity $n - 1$. And since the trace of $M_n$ is $3n$ (which is the sum of the eigenvalues), the last eigenvalue must be $3n$. The
eigenvectors corresponding to $\lambda = 0$ are the ones that sum to zero, so a basis is

$$
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -1
\end{bmatrix}
$$

and the eigenvector corresponding to $\lambda = 3n$ is

$$
\begin{bmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
$$

(d) From part (c) we see that

$$
D = \begin{bmatrix}
3n & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
$$

and

$$
P = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -1
\end{bmatrix}
$$

(e) Let $U_n$ be the $n \times n$ matrix with all 1’s in it, so $M_n = 3U_n$. The $U_{n}^2 = nU_n$, $U_{n}^3 = n^2U_n$, etc., so that $U_{n}^k = n^{k-1}U_n$. Therefore $M_{n}^k = (3U_n)^k = 3^kn^{k-1}U_n$. Therefore

$$
e^{tM_n} = I + tM_n + \frac{t^2M_n^2}{2!} + \frac{t^3M_n^3}{3!} + \cdots + \frac{t^kM_n^k}{k!} + \cdots
$$

$$
= I + 3tU_n + \frac{3^2t^2nU_n}{2!} + \frac{3^3t^3n^2U_n}{3!} + \cdots + \frac{3^ktn^kU_n}{k!} + \cdots
$$

$$
= I + \frac{1}{n}U_n \left( 3tn + \frac{(3tn)^2}{2!} + \frac{(3tn)^3}{3!} + \cdots + \frac{(3tn)^k}{k!} + \cdots \right)
$$

$$
= I + \frac{e^{3tn} - 1}{n}U_n
$$

$$
= \begin{bmatrix}
1 + \frac{1}{n}(e^{3tn} - 1) & \frac{1}{n}(e^{3tn} - 1) & \frac{1}{n}(e^{3tn} - 1) & \cdots & \frac{1}{n}(e^{3tn} - 1) \\
\frac{1}{n}(e^{3tn} - 1) & 1 + \frac{1}{n}(e^{3tn} - 1) & \frac{1}{n}(e^{3tn} - 1) & \cdots & \frac{1}{n}(e^{3tn} - 1) \\
\frac{1}{n}(e^{3tn} - 1) & \frac{1}{n}(e^{3tn} - 1) & 1 + \frac{1}{n}(e^{3tn} - 1) & \cdots & \frac{1}{n}(e^{3tn} - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n}(e^{3tn} - 1) & \frac{1}{n}(e^{3tn} - 1) & \frac{1}{n}(e^{3tn} - 1) & \cdots & 1 + \frac{1}{n}(e^{3tn} - 1)
\end{bmatrix}
$$
4. Let \( M_n \) be the \( n \times n \) matrix all of whose entries are 5. So for example:

\[
M_4 = \begin{bmatrix}
5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5 \\
5 & 5 & 5 & 5
\end{bmatrix}.
\]

(If you are uncomfortable dealing with \( M_n \) for arbitrary \( n \), you can do the \( n = 4 \) case and receive most of the credit for this problem.)

(a) What is \( \text{tr}(M_n) \)?

(b) What is \( \text{det}(M_n) \)?

(c) What are the eigenvalues and eigenvectors of \( M_n \)?

(d) Find a diagonal matrix \( D \) and an invertible matrix \( P \) so that \( M_n = PD P^{-1} \).

(e) \textit{Extra credit}: What is \( e^{tM_n} \)?

(a) \( \text{tr}(M_n) = 5n \)

(b) \( \text{det}(M_n) = 0 \) since there are many identical rows.

(c) Since \( M_n \) has rank 1, \( \lambda = 0 \) is an eigenvalue with multiplicity \( n - 1 \). And since the trace of \( M_n \) is \( 5n \) (which is the sum of the eigenvalues), the last eigenvalue must be \( 5n \). The eigenvectors corresponding to \( \lambda = 0 \) are the ones that sum to zero, so a basis is

\[
\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ -1 \end{bmatrix} \right\}
\]

and the eigenvector corresponding to \( \lambda = 5n \) is

\[
\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
\]
(d) From part (c) we see that

\[
D = \begin{bmatrix}
5n & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\end{bmatrix}
\quad \text{and} \quad
P = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & -1 & 0 & 0 & \cdots & 0 \\
1 & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -1 \\
\end{bmatrix}
\]

(e) Let \(U_n\) be the \(n \times n\) matrix with all 1’s in it, so \(M_n = 5U_n\). The \(U_2^n = nU_n\), \(U_3^n = n^2U_n\), etc., so that \(U_k^n = n^{k-1}U_n\). Therefore \(M_k^n = (5U_n)^k = 5^k n^{k-1}U_n\). Therefore

\[
e^{tM_n} = I + tM_n + \frac{t^2 M_n^2}{2!} + \frac{t^3 M_n^3}{3!} + \cdots + \frac{t^k M_n^k}{k!} + \cdots \\
= I + 5tU_n + \frac{5^2 t^2 nU_n}{2!} + \frac{5^3 t^3 n^2U_n}{3!} + \cdots + \frac{5^k t^k n^{k-1}U_n}{k!} + \cdots \\
= I + \frac{1}{n} U_n \left(5tn + \frac{(5tn)^2}{2!} + \frac{(5tn)^3}{3!} + \cdots + \frac{(5tn)^k}{k!} + \cdots \right) \\
= I + \frac{e^{5tn} - 1}{n} U_n
\]

\[
= \begin{bmatrix}
1 + \frac{1}{n} (e^{5tn} - 1) & \frac{1}{n} (e^{5tn} - 1) & \frac{1}{n} (e^{5tn} - 1) & \cdots & \frac{1}{n} (e^{5tn} - 1) \\
\frac{1}{n} (e^{5tn} - 1) & 1 + \frac{1}{n} (e^{5tn} - 1) & \frac{1}{n} (e^{5tn} - 1) & \cdots & \frac{1}{n} (e^{5tn} - 1) \\
\frac{1}{n} (e^{5tn} - 1) & \frac{1}{n} (e^{5tn} - 1) & 1 + \frac{1}{n} (e^{5tn} - 1) & \cdots & \frac{1}{n} (e^{5tn} - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n} (e^{5tn} - 1) & \frac{1}{n} (e^{5tn} - 1) & \frac{1}{n} (e^{5tn} - 1) & \cdots & 1 + \frac{1}{n} (e^{5tn} - 1)
\end{bmatrix}
\]

4. Let \(M_n\) be the \(n\)-by-\(n\) matrix all of whose entries are 6. So for example:

\[
M_4 = \begin{bmatrix}
6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6 \\
6 & 6 & 6 & 6
\end{bmatrix}
\]

(If you are uncomfortable dealing with \(M_n\) for arbitrary \(n\), you can do the \(n = 4\) case and receive most of the credit for this problem.)

(a) What is \(\text{tr}(M_n)\)?

(b) What is \(\text{det}(M_n)\)?

(c) What are the eigenvalues and eigenvectors of \(M_n\)?
(d) Find a diagonal matrix $D$ and an invertible matrix $P$ so that $M_n = PDP^{-1}$.

(e) Extra credit: What is $e^{tM_n}$?

(a) $\text{tr}(M_n) = 6n$

(b) $\det(M_n) = 0$ since there are many identical rows.

(c) Since $M_n$ has rank 1, $\lambda = 0$ is an eigenvalue with multiplicity $n - 1$. And since the trace of $M_n$ is $6n$ (which is the sum of the eigenvalues), the last eigenvalue must be $6n$. The eigenvectors corresponding to $\lambda = 0$ are the ones that sum to zero, so a basis is

$$
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

and the eigenvector corresponding to $\lambda = 6n$ is

$$
\begin{bmatrix}
1 \\
1 \\
1 \\
\vdots \\
1
\end{bmatrix}.
$$

(d) From part (c) we see that

$$
D = \begin{bmatrix}
6n & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
P = \begin{bmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & -1 & 0 & 0 & \cdots & 0 \\
1 & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}.
$$

(e) Let $U_n$ be the $n \times n$ matrix with all 1’s in it, so $M_n = 6U_n$. The $U_n^2 = nU_n$, $U_n^3 = n^2U_n$, 

$$
U_n^4 = n^3U_n,
$$

and so on.
etm = I + tM_n + \frac{t^2 M_n^2}{2!} + \frac{t^3 M_n^3}{3!} + \ldots + \frac{t^k M_n^k}{k!} + \ldots

= I + 6tU_n + \frac{6^2 t^2 n U_n}{2!} + \frac{6^3 t^3 n^2 U_n}{3!} + \ldots + \frac{6^k t^k n^k U_n}{k!} + \ldots

= I + \frac{1}{n} U_n \left( 6tn + \frac{(6tn)^2}{2!} + \frac{(6tn)^3}{3!} + \ldots + \frac{(6tn)^k}{k!} + \ldots \right)

= I + \frac{e^{6tn} - 1}{n} U_n

4. Let M_n be the n-by-n matrix all of whose entries are 7. So for example:

M_4 = \begin{bmatrix} 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \\ 7 & 7 & 7 & 7 \end{bmatrix}

(If you are uncomfortable dealing with M_n for arbitrary n, you can do the n = 4 case and receive most of the credit for this problem.)

(a) What is tr(M_n)?

(b) What is det(M_n)?

(c) What are the eigenvalues and eigenvectors of M_n?

(d) Find a diagonal matrix D and an invertible matrix P so that M_n = PDP^{-1}.

(e) Extra credit: What is e^{tM_n}?

(a) tr(M_n) = 7n

(b) det(M_n) = 0 since there are many identical rows.

(c) Since M_n has rank 1, λ = 0 is an eigenvalue with multiplicity n - 1. And since the trace of M_n is 7n (which is the sum of the eigenvalues), the last eigenvalue must be 7n. The
eigenvectors corresponding to \( \lambda = 0 \) are the ones that sum to zero, so a basis is
\[
\begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
-1 & 0 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\]
and the eigenvector corresponding to \( \lambda = 7n \) is
\[
\begin{bmatrix}
1 \\
1 \\
\vdots \\
1
\end{bmatrix}
\]

(d) From part (c) we see that
\[
D = \begin{bmatrix}
7n & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{bmatrix}
\quad \text{and} \quad
P = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 \\
1 & -1 & 0 & 0 & \cdots & 0 \\
1 & 0 & -1 & 0 & \cdots & 0 \\
1 & 0 & 0 & -1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & -1
\end{bmatrix}
\]

(e) Let \( U_n \) be the \( n \times n \) matrix with all 1’s in it, so \( M_n = 7U_n \). The \( U_n^2 = nU_n \), \( U_n^3 = n^2U_n \), etc., so that \( U_n^k = n^{k-1}U_n \). Therefore \( M_n^k = (7U_n)^k = 7^kn^{k-1}U_n \). Therefore
\[
e^{tM_n} = I + tM_n + \frac{t^2M_n^2}{2!} + \frac{t^3M_n^3}{3!} + \cdots + \frac{t^kM_n^k}{k!} + \cdots
\]
\[
= I + 7tU_n + \frac{7^2t^2nU_n}{2!} + \frac{7^3t^3n^2U_n}{3!} + \cdots + \frac{7^kt^kn^{k-1}U_n}{k!} + \cdots
\]
\[
= I + \frac{1}{n}U_n \left( 7tn + \frac{(7tn)^2}{2!} + \frac{(7tn)^3}{3!} + \cdots + \frac{(7tn)^k}{k!} + \cdots \right)
\]
\[
= I + \frac{e^{7tn} - 1}{n}U_n
\]
\[
= \begin{bmatrix}
1 + \frac{1}{n}(e^{7tn} - 1) & \frac{1}{n}(e^{7tn} - 1) & \frac{1}{n}(e^{7tn} - 1) & \cdots & \frac{1}{n}(e^{7tn} - 1) \\
\frac{1}{n}(e^{7tn} - 1) & 1 + \frac{1}{n}(e^{7tn} - 1) & \frac{1}{n}(e^{7tn} - 1) & \cdots & \frac{1}{n}(e^{7tn} - 1) \\
\frac{1}{n}(e^{7tn} - 1) & \frac{1}{n}(e^{7tn} - 1) & 1 + \frac{1}{n}(e^{7tn} - 1) & \cdots & \frac{1}{n}(e^{7tn} - 1) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{n}(e^{7tn} - 1) & \frac{1}{n}(e^{7tn} - 1) & \frac{1}{n}(e^{7tn} - 1) & \cdots & 1 + \frac{1}{n}(e^{7tn} - 1)
\end{bmatrix}
\]
7. Let 

\[ A = \begin{bmatrix} 5 & 12 \\ -2 & -5 \end{bmatrix}. \]

(a) Find the eigenvalues and eigenvectors of \( A \)

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) so that \( P^{-1}AP = D \).

(c) Calculate \( e^{tA} \)

(a) The characteristic polynomial of \( A \) is \((5 - \lambda)(-5 - \lambda) + 24 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)\), so the eigenvalues of \( A \) are \( \lambda = \pm 1 \). For \( \lambda = 1 \) we need the kernel of

\[ \begin{bmatrix} 4 & 12 \\ -2 & -6 \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \)

and for \( \lambda = -1 \) we need the kernel of

\[ \begin{bmatrix} 6 & 12 \\ -2 & -4 \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \)

(b) \( P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}. \)

(c) Since \( \det P = 1 \), \( P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \). So

\[ e^{tA} = Pe^{tD}P^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2e^{-t} & -3e^t \\ e^{-t} & e^t \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2e^{-t} + 6e^t & -3e^{-t} + 6e^t \\ e^{-t} - e^t & 3e^{-t} - 2e^t \end{bmatrix} \]

5. Let 

\[ A = \begin{bmatrix} -5 & -12 \\ 2 & 5 \end{bmatrix}. \]

(a) Find the eigenvalues and eigenvectors of \( A \)

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) so that \( P^{-1}AP = D \).

(c) Calculate \( e^{tA} \)
(a) The characteristic polynomial of $A$ is $(-5 - \lambda)(5 - \lambda) + 24 = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1)$, so the eigenvalues of $A$ are $\lambda = \pm 1$. For $\lambda = 1$ we need the kernel of

$$\begin{bmatrix} -6 & -12 \\ 2 & 4 \end{bmatrix}$$

which is spanned by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$, and for $\lambda = -1$ we need the kernel of

$$\begin{bmatrix} -4 & -12 \\ 2 & 6 \end{bmatrix}$$

which is spanned by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$.

(b)

$$P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

(c) Since $\det P = 1$, $P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}$. So

$$e^{tA} = Pe^{tD}P^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2e^t & -3e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2e^t + 3e^{-t} & -6e^t + 6e^{-t} \\ e^t - e^{-t} & 3e^t - 2e^{-t} \end{bmatrix}$$

5. Let

$$A = \begin{bmatrix} 7 & 18 \\ -3 & -8 \end{bmatrix}.$$ 

(a) Find the eigenvalues and eigenvectors of $A$

(b) Find an invertible matrix $P$ and a diagonal matrix $D$ so that $P^{-1}AP = D$.

(c) Calculate $e^{tA}$

(a) The characteristic polynomial of $A$ is $(7 - \lambda)(-8 - \lambda) + 54 = \lambda^2 + \lambda - 2 = (\lambda - 1)(\lambda + 2)$, so the eigenvalues of $A$ are $\lambda = 1$ and $\lambda = -2$. For $\lambda = 1$ we need the kernel of

$$\begin{bmatrix} 6 & 18 \\ -3 & -9 \end{bmatrix}$$

which is spanned by $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and for $\lambda = -2$ we need the kernel of

$$\begin{bmatrix} 9 & 18 \\ -3 & -6 \end{bmatrix}$$

which is spanned by $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.
(b) \[ P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}. \]

(c) Since \( \det P = 1 \), \( P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \). So
\[
e^{tA} = P e^{tD} P^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2e^{-2t} & -3e^t \\ e^{-2t} & e^t \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}
\]
\[
= \begin{bmatrix} -2e^{-2t} + 3e^t & -6e^{-2t} + 6e^t \\ e^{-2t} - e^t & 3e^{-2t} - 2e^t \end{bmatrix}
\]

5. Let \( A = \begin{bmatrix} 8 & 18 \\ -3 & -7 \end{bmatrix} \).

(a) Find the eigenvalues and eigenvectors of \( A \)

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) so that \( P^{-1}AP = D \).

(c) Calculate \( e^{tA} \)

(a) The characteristic polynomial of \( A \) is \( (8-\lambda)(-7-\lambda) + 54 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) \), so the eigenvalues of \( A \) are \( \lambda = 2 \) and \( \lambda = -1 \). For \( \lambda = 2 \) we need the kernel of
\[
\begin{bmatrix} 6 & 18 \\ -3 & -9 \end{bmatrix}
\]
which is spanned by \( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \)

and for \( \lambda = -1 \) we need the kernel of
\[
\begin{bmatrix} 9 & 18 \\ -3 & -6 \end{bmatrix}
\]
which is spanned by \( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \)

(b) \[ P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}. \]

(c) Since \( \det P = 1 \), \( P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \). So
\[
e^{tA} = P e^{tD} P^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2e^{-t} & -3e^{2t} \\ e^{-t} & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix}
\]
\[
= \begin{bmatrix} -2e^{-t} + 3e^{2t} & -6e^{-t} + 6e^{2t} \\ e^{-t} - e^{2t} & 3e^{-t} - 2e^{2t} \end{bmatrix}
\]
5. Let 

\[ A = \begin{bmatrix} 10 & 24 \\ -4 & -10 \end{bmatrix}. \]

(a) Find the eigenvalues and eigenvectors of \( A \)

(b) Find an invertible matrix \( P \) and a diagonal matrix \( D \) so that \( P^{-1}AP = D \).

(c) Calculate \( e^{tA} \)

(a) The characteristic polynomial of \( A \) is \( (10 - \lambda)(-10 - \lambda) + 96 = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2) \), so the eigenvalues of \( A \) are \( \lambda = \pm 2 \). For \( \lambda = 2 \) we need the kernel of 

\[ \begin{bmatrix} 8 & 24 \\ -4 & -12 \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} -3 \\ 1 \end{bmatrix} \)

and for \( \lambda = -2 \) we need the kernel of 

\[ \begin{bmatrix} 12 & 24 \\ -4 & -8 \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} -2 \\ 1 \end{bmatrix} \)

(b) 

\[ P = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}. \]

(c) Since \( \det P = 1 \), \( P^{-1} = \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \). So

\[ e^{tA} = Pe^{tD}P^{-1} = \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2e^{-2t} & -3e^{2t} \\ e^{-2t} & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} \]

\[ = \begin{bmatrix} -2e^{-2t} + 3e^{2t} & -6e^{-2t} + 6e^{2t} \\ e^{-2t} - e^{2t} & 3e^{-2t} - 2e^{2t} \end{bmatrix} \]

6. **True or false**: For each statement, say whether it is true (meaning always true), or false (meaning it could be true under certain special conditions but it is not always true). For the FALSE statements only, give a reason or an example to show that they are false.

(a) All upper triangular matrices are diagonalizable.

False. \( \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \) is not.

(a) All lower triangular matrices are diagonalizable.
(a) All invertible matrices are diagonalizable.

False. \[
\begin{bmatrix}
  3 & 0 \\
  1 & 3
\end{bmatrix}
\] is not.

(b) If \( b > 0 \) and \( c > 0 \), then all solutions of \( y'' + by' + cy = 0 \) approach zero as \( t \to \infty \)

True. The roots of \( r^2 + br + c = 0 \) are \( r = \frac{-b \pm \sqrt{b^2 - 4c}}{2} \). Since \( b > 0 \), the real part of this is \(-b/2\) and hence negative if \( 4c^2 > b \), and if the roots are both real they are negative since \( b^2 - 4c < b^2 \) since \( c > 0 \).

(b) If \( b > 0 \) and \( c > 0 \), then all solutions of \( y'' + by' + cy = 0 \) oscillate infinitely often.

False. See above for the roots, if \( 4c \leq b^2 \) then the roots are real so the solutions do not oscillate.

(c) If the characteristic polynomial of a 3-by-3 matrix is \(-\lambda^3 + 2\lambda^2 - \lambda + 5\) then the matrix is invertible

True. Zero is not an eigenvalue, since the value of the polynomial at \( \lambda = 0 \) is 5.

(c) If the characteristic polynomial of a 3-by-3 matrix is \(-\lambda^3 - 2\lambda^2 + \lambda\) then the matrix is invertible

False. Zero is an eigenvalue.

(c) If the characteristic polynomial of a 3-by-3 matrix is \(-\lambda^3 - \lambda + 1\) then the matrix is invertible

True. Zero is not an eigenvalue.

(d) If the determinant of a 2-by-2 matrix is 9 and its trace is 6 then the matrix is diagonalizable.

False – the sum of the eigenvalues is 6 and their product is 9, so the eigenvalues are 3 and 3. The matrix could be \[
\begin{bmatrix}
  3 & 1 \\
  0 & 3
\end{bmatrix}
\] which is not diagonalizable.
(d) If the determinant of a 2-by-2 matrix is 9 and its trace is \(-6\) then the matrix is diagonalizable.

False. Eigenvalues are \(-3\) and \(-3\). The matrix could be \[
\begin{bmatrix}
-3 & 1 \\
0 & -3 \\
\end{bmatrix}
\] which is not diagonalizable.

(d) If the determinant of a 2-by-2 matrix is 9 and its trace is 10 then the matrix is diagonalizable.

True. The eigenvalues are 1 and 9.

(d) If the determinant of a 2-by-2 matrix is 9 and its trace is \(-10\) then the matrix is diagonalizable.

True. The eigenvalues are \(-1\) and \(-9\).
1. (a) Find the general solution to the system of differential equations:

\[ x' = 2x + y \quad y' = -x + 2y \]

This is \( x' = Ax \) with \( A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \). The characteristic polynomial of \( A \) is \( \lambda^2 - 4\lambda + 5 \). The eigenvalues are \( \lambda = 2 \pm i \). For \( \lambda = 2 + i \) we need the kernel of

\[ \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} 1 \\ i \end{bmatrix} \)

So we need the real and imaginary parts of

\[ e^{(2+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{2t}(\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{2t} \cos t \\ -e^{2t} \sin t \end{bmatrix} + i \begin{bmatrix} e^{2t} \sin t \\ e^{2t} \cos t \end{bmatrix} \]

So the general solution of the system is

\[ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} \cos t + c_2 e^{2t} \sin t \\ -c_1 e^{2t} \sin t + c_2 e^{2t} \cos t \end{bmatrix} \]

(b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions

\[ x(0) = 0, \quad y(0) = 2 \]

and draw a graph of the trajectory of the point \((x(t), y(t))\) in the \(xy\)-plane. Put an arrow on the curve to indicate the direction of increasing \(t\).

Since \( x(0) = c_1 \) and \( y(0) = c_2 \) in our general solution above, we have \( c_1 = 0 \) and \( c_2 = 2 \). So the solution is

\[ x = 2e^{2t} \sin t \quad \text{and} \quad y = 2e^{2t} \cos t. \]

The graph of this is a clockwise spiral out:

1. (a) Find the general solution to the system of differential equations:

\[ x' = -2x + y \quad y' = -x - 2y \]

This is \( x' = Ax \) with \( A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \). The characteristic polynomial of \( A \) is \( -2 - \lambda(-2 - \lambda) + 1 = \lambda^2 + 4\lambda + 5 \). The eigenvalues are \( \lambda = -2 \pm i \). For \( \lambda = -2 + i \) we need the kernel of

\[ \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \]

which is spanned by \( \begin{bmatrix} 1 \\ i \end{bmatrix} \)
So we need the real and imaginary parts of
\[ e^{(-2+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{-2t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^{-2t} \cos t \\ -e^{-2t} \sin t \end{bmatrix} + i \begin{bmatrix} e^{-2t} \sin t \\ e^{2t} \cos t \end{bmatrix} \]

So the general solution of the system is
\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t \\ -c_1 e^{-2t} \sin t + c_2 e^{-2t} \cos t \end{bmatrix}
\]

(b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions
\[ x(0) = 0, \quad y(0) = 2 \]
and draw a graph of the trajectory of the point \((x(t), y(t))\) in the \(xy\)-plane. Put an arrow on the curve to indicate the direction of increasing \(t\).

Since \(x(0) = c_1\) and \(y(0) = c_2\) in our general solution above, we have \(c_1 = 0\) and \(c_2 = 2\). So the solution is
\[ x = 2e^{-2t} \sin t \quad \text{and} \quad y = 2e^{-2t} \cos t. \]

The graph of this is a clockwise spiral in:

1. (a) Find the general solution to the system of differential equations:
\[ x' = 3x - y \quad y' = x + 3y \]

This is \(\mathbf{x}' = A\mathbf{x}\) with \(A = \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\). The characteristic polynomial of \(A\) is \((3 - \lambda)(3 - \lambda) + 1 = \lambda^2 - 6\lambda + 10\). The eigenvalues are \(\lambda = 3 + i\). For \(\lambda = 3 + i\) we need the kernel of
\[
\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}
\]
which is spanned by \(\begin{bmatrix} 1 \\ -i \end{bmatrix}\)

So we need the real and imaginary parts of
\[ e^{(3+i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{3t} (\cos t + i \sin t) \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^{3t} \cos t \\ e^{3t} \sin t \end{bmatrix} + i \begin{bmatrix} e^{3t} \sin t \\ -e^{3t} \cos t \end{bmatrix} \]

So the general solution of the system is
\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} \cos t + c_2 e^{3t} \sin t \\ c_1 e^{3t} \sin t - c_2 e^{3t} \cos t \end{bmatrix}
\]

(b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions
\[ x(0) = 0, \quad y(0) = 2 \]
and draw a graph of the trajectory of the point \((x(t), y(t))\) in the \(xy\)-plane. Put an arrow on the curve to indicate the direction of increasing \(t\).

Since \(x(0) = c_1\) and \(y(0) = -c_2\) in our general solution above, we have \(c_1 = 0\) and \(c_2 = -2\). So the solution is

\[
\begin{align*}
x &= -2e^{3t}\sin t \\
y &= 2e^{3t}\cos t.
\end{align*}
\]

The graph of this is a counterclockwise spiral out:

1. (a) Find the general solution to the system of differential equations:

\[
\begin{align*}
x' &= -3x - y \\
y' &= x - 3y
\end{align*}
\]

This is \(x' = Ax\) with \(A = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}\). The characteristic polynomial of \(A\) is \((-3 - \lambda)(-3 - \lambda) + 1 = \lambda^2 + 6\lambda + 10\). The eigenvalues are \(\lambda = -3 \pm i\). For \(\lambda = -3 + i\) we need the kernel of

\[
\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}
\]

which is spanned by \(\begin{bmatrix} 1 \\ -i \end{bmatrix}\)

So we need the real and imaginary parts of

\[
e^{(-3+i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix} = e^{-3t}(\cos t + i \sin t)\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} e^{-3t}\cos t \\ e^{-3t}\sin t \end{bmatrix} + i\begin{bmatrix} e^{-3t}\sin t \\ -e^{-3t}\cos t \end{bmatrix}
\]

So the general solution of the system is

\[
\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1e^{-3t}\cos t + c_2e^{-3t}\sin t \\ c_1e^{-3t}\sin t - c_2e^{-3t}\cos t \end{bmatrix}
\]

(b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions

\[
x(0) = 0, \quad y(0) = 2
\]

and draw a graph of the trajectory of the point \((x(t), y(t))\) in the \(xy\)-plane. Put an arrow on the curve to indicate the direction of increasing \(t\).

Since \(x(0) = c_1\) and \(y(0) = -c_2\) in our general solution above, we have \(c_1 = 0\) and \(c_2 = -2\). So the solution is

\[
\begin{align*}
x &= -2e^{-3t}\sin t \\
y &= 2e^{-3t}\cos t.
\end{align*}
\]

The graph of this is a counterclockwise spiral in:

1. (a) Find the general solution to the system of differential equations:

\[
\begin{align*}
x' &= x + 2y \\
y' &= -2x + y
\end{align*}
\]
This is $x' = Ax$ with $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. The characteristic polynomial of $A$ is $(1 - \lambda)(1 - \lambda) + 4 = \lambda^2 - 2\lambda + 5$. The eigenvalues are $\lambda = 1 \pm 2i$. For $\lambda = 1 + 2i$ we need the kernel of $\begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix}$ which is spanned by $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

So we need the real and imaginary parts of $e^{(1+2i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^t (\cos 2t + i \sin 2t) \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix} + i \begin{bmatrix} e^t \sin 2t \\ e^t \cos 2t \end{bmatrix}$.

So the general solution of the system is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} c_1e^t \cos 2t + c_2e^t \sin 2t \\ -c_1e^t \sin 2t + c_2e^t \cos 2t \end{bmatrix}$.

(b) Find the unique solution of the differential system in part (a) that also satisfies the initial conditions $x(0) = 0$, $y(0) = 2$ and draw a graph of the trajectory of the point $(x(t), y(t))$ in the $xy$-plane. Put an arrow on the curve to indicate the direction of increasing $t$.

Since $x(0) = c_1$ and $y(0) = c_2$ in our general solution above, we have $c_1 = 0$ and $c_2 = 2$. So the solution is $x = 2e^t \sin 2t$ and $y = 2e^t \cos 2t$.

The graph of this is a clockwise spiral out:

2. Find the general solution of the system $x' = Ax$, where $A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

Since $A$ is upper-triangular, its eigenvalues are its diagonal entries, namely $\lambda = 3, 2, 2$.

For $\lambda = 3$ we need the kernel of $A - 3I = \begin{bmatrix} 0 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ which is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

and for $\lambda = 2$ we need the kernel of $A - 2I = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ which is spanned by $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
Since there is only one linearly independent eigenvector for the double eigenvalue $\lambda = 2$, we need a generalized eigenvector, i.e., a vector $w$ for which $(A - 2I)w = v$. One such vector is

$$w = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

so we can write the general solution of $x' = Ax$ as

$$x = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}$$

where the third vector is $tv + w$.

2. Find the general solution of the system $x' = Ax$, where

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Since $A$ is upper-triangular, its eigenvalues are its diagonal entries, namely $\lambda = 2, 3, 3$.

For $\lambda = 2$ we need the kernel of

$$A - 2I = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is spanned by $v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and for $\lambda = 3$ we need the kernel of

$$A - 3I = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is spanned by $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Since there is only one linearly independent eigenvector for the double eigenvalue $\lambda = 3$, we need a generalized eigenvector, i.e., a vector $w$ for which $(A - 3I)w = v$. One such vector is

$$w = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

so we can write the general solution of $x' = Ax$ as

$$x = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{3t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}$$
where the third vector is $tv + w$.

2. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 6 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Since $A$ is upper-triangular, its eigenvalues are its diagonal entries, namely $\lambda = 1, -2, -2$.

For $\lambda = 1$ we need the kernel of

$$A - I = \begin{bmatrix} 0 & 0 & 6 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

which is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and for $\lambda = -2$ we need the kernel of

$$A + 2I = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is spanned by $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Since there is only one linearly independent eigenvector for the double eigenvalue $\lambda = -2$, we need a generalized eigenvector, i.e., a vector $\mathbf{w}$ for which $(A + 2I)\mathbf{w} = \mathbf{v}$. One such vector is

$$\mathbf{w} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},$$

so we can write the general solution of $\mathbf{x}' = A\mathbf{x}$ as

$$\mathbf{x} = c_1 e^t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{-2t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}$$

where the third vector is $tv + w$.

2. Find the general solution of the system $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} -1 & 0 & -6 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Since $A$ is upper-triangular, its eigenvalues are its diagonal entries, namely $\lambda = -1, 2, 2$. 

For $\lambda = -1$ we need the kernel of
\[
A + I = \begin{bmatrix}
0 & 0 & -6 \\
0 & 3 & 1 \\
0 & 0 & 3
\end{bmatrix}
\]
which is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and for $\lambda = 2$ we need the kernel of
\[
A - 2I = \begin{bmatrix}
-3 & 0 & -6 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
which is spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Since there is only one linearly independent eigenvector for the double eigenvalue $\lambda = 2$, we need a generalized eigenvector, i.e., a vector $w$ for which $(A - 2I)w = v$. One such vector is $w = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, so we can write the general solution of $x' = Ax$ as
\[
x = c_1 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}
\]
where the third vector is $tv + w$.

---

**2.** Find the general solution of the system $x' = Ax$, where
\[
A = \begin{bmatrix}
3 & 0 & 4 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

Since $A$ is upper-triangular, its eigenvalues are its diagonal entries, namely $\lambda = 3, 1, 1$.

For $\lambda = 3$ we need the kernel of
\[
A - 3I = \begin{bmatrix}
0 & 0 & 4 \\
0 & -2 & 1 \\
0 & 0 & -2
\end{bmatrix}
\]
which is spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and for $\lambda = 1$ we need the kernel of
\[
A - I = \begin{bmatrix}
2 & 0 & 4 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
which is spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.
Since there is only one linearly independent eigenvector for the double eigenvalue \( \lambda = 1 \), we need a generalized eigenvector, i.e., a vector \( \mathbf{w} \) for which \((A - I)\mathbf{w} = \mathbf{v}\). One such vector is

\[
\mathbf{w} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix},
\]

so we can write the general solution of \( \mathbf{x}' = A\mathbf{x} \) as

\[
\mathbf{x} = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^t \begin{bmatrix} -2 \\ t \\ 1 \end{bmatrix}
\]

where the third vector is \( t\mathbf{v} + \mathbf{w} \).

3. For each of the following matrices \( A \), graph the solutions of \( \mathbf{x}' = A\mathbf{x} \) in the phase plane. Give enough reasoning so that I can give you partial credit if you get it wrong because of some arithmetic error.

(a) \[
\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

\( \det < 0 \) so saddle – out along \( y = x \), in along \( y = -x \).

Figure 1: Solutions of \( x' = y, \ y' = x \).
(b) \[
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\]
det > 0 and tr = 0 so center \(-x' > 0\) for \(y > 0\) so clockwise circles

Figure 2: Solutions of \(x' = y, y' = -x\).

(c) \[
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]
det > 0 and tr = 0 so center \(-x' < -0\) for \(y > 0\) so counterclockwise circles

Figure 3: Solutions of \(x' = -y, y' = x\).

(d) \[
\begin{bmatrix}
3 & 1 \\
1 & 3
\end{bmatrix}
\]
det > 0 and tr > 0 but \(tr^2 > 4 \text{det}\) so a source. Tangent to \(x = -y\), parallel to \(x = y\)
Figure 4: Solutions of \( x' = 3x + y, \ y' = x + 3y. \)

3. For each of the following matrices \( A \), graph the solutions of \( \mathbf{x}' = A\mathbf{x} \) in the phase plane. Give enough reasoning so that I can give you partial credit if you get it wrong because of some arithmetic error.

(a) \[
\begin{bmatrix}
0 & -2 \\
2 & 0
\end{bmatrix}
\]

\( \text{det} > 0 \) and \( \text{tr} = 0 \) so center – \( \mathbf{x}' < -0 \) for \( y > 0 \) so counterclockwise circles

Figure 5: Solutions of \( x' = -2y, \ y' = 2x. \)
(b) $\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

$\det < 0$ so saddle – in along $y = x$, out along $y = -x$.

Figure 6: Solutions of $x' = -2y$, $y' = -2x$.

(c) $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

$\det > 0$ and $\text{tr} = 0$ so center – $x' > 0$ for $y > 0$ so clockwise circles

Figure 7: Solutions of $x' = 2y$, $y' = -2x$. 
(d) \[
\begin{bmatrix}
3 & -1 \\
-1 & 3 \\
\end{bmatrix}
\]

$\det > 0$ and $\text{tr} > 0$ but $\text{tr}^2 > 4 \det$ so a source. Tangent to $x = y$, parallel to $x = -y$

Figure 8: Solutions of $x' = 3x - y$, $y' = -x + 3y$.

---

3. For each of the following matrices $A$, graph the solutions of $\mathbf{x}' = A\mathbf{x}$ in the phase plane. Give enough reasoning so that I can give you partial credit if you get it wrong because of some arithmetic error.

(a) \[
\begin{bmatrix}
0 & 3 \\
-3 & 0 \\
\end{bmatrix}
\]

$\det > 0$ and $\text{tr} = 0$ so center $-x' > 0$ for $y > 0$ so clockwise circles

Figure 9: Solutions of $x' = 3y$, $y' = -3x$.

(b) \[
\begin{bmatrix}
0 & -3 \\
3 & 0 \\
\end{bmatrix}
\]

$\det > 0$ and $\text{tr} = 0$ so center $-x' < 0$ for $y > 0$ so counterclockwise circles
Figure 10: Solutions of $x' = -3y$, $y' = 3x$.

$$
\begin{pmatrix}
0 & 3 \\
3 & 0
\end{pmatrix}
$$

$\det < 0$ so saddle-out along $y = x$, in along $y = -x$.

Figure 11: Solutions of $x' = 3y$, $y' = 3x$. 
3. For each of the following matrices $A$, graph the solutions of $\mathbf{x}' = A\mathbf{x}$ in the phase plane. Give enough reasoning so that I can give you partial credit if you get it wrong because of some arithmetic error.

(a) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

det < 0 so saddle – in along $y = x$, out along $y = -x$.

(b) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

det > 0 and tr = 0 so center – $\mathbf{x}' < -0$ for $y > 0$ so counterclockwise circles
Figure 14: Solutions of $x' = -y$, $y' = x$.

(c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\det > 0$ and $\text{tr} = 0$ so center $x' > 0$ for $y > 0$ so clockwise circles

Figure 15: Solutions of $x' = y$, $y' = -x$. 
(d) \[
\begin{bmatrix}
-3 & 1 \\
1 & -3
\end{bmatrix}
\]
det > 0 and tr < 0 but tr^2 > 4 det so a source. Tangent to x = y, parallel to x = -y

Figure 16: Solutions of \( x' = -3x + y, \ y' = x - 3y \).

3. For each of the following matrices \( A \), graph the solutions of \( x' = Ax \) in the phase plane. Give enough reasoning so that I can give you partial credit if you get it wrong because of some arithmetic error.

(a) \[
\begin{bmatrix}
0 & 2 \\
2 & 0
\end{bmatrix}
\]
det < 0 so saddle - out along \( y = x \), in along \( y = -x \).

Figure 17: Solutions of \( x' = 2y, \ y' = 2x \).

(b) \[
\begin{bmatrix}
0 & -2 \\
2 & 0
\end{bmatrix}
\]
det > 0 and tr = 0 so center - \( x' < -0 \) for \( y > 0 \) so counterclockwise circles
Figure 18: Solutions of $x' = -2y$, $y' = 2x$.

(c) $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

det > 0 and tr = 0 so center $x' > 0$ for $y > 0$ so clockwise circles

Figure 19: Solutions of $x' = 2y$, $y' = -2x$. 
(d) \[
\begin{bmatrix}
1 & -3 \\
-3 & 1
\end{bmatrix}
\]

\[\text{det} < 0 \text{ and } \text{tr} > 0 \text{ so saddle-out along } y = -x, \text{ in along } y = x.\]

Figure 20: Solutions of \(x' = x - 3y, y' = -3x + y\).

---

4. Suppose \(x' = Ax\), where

\[
x = \begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t) \\
x_4(t) \\
x_5(t)
\end{bmatrix}
\quad \text{and} \quad
A = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 2 & 1 & 0 & 0 \\
0 & 0 & 3 & 1 & 0 \\
0 & 0 & 0 & 4 & 1 \\
0 & 0 & 0 & 0 & 5
\end{bmatrix}
\]

and \(x\) satisfies the initial condition

\[
x(0) = \begin{bmatrix}
5 \\
10 \\
20 \\
30 \\
24
\end{bmatrix}
\]

What is \(x_1(t)\)?

It might be easiest to do this one by "cleverness". Note that the last equation says \(x'_5 = 5x_5\) with \(x_5(0) = 24\). Therefore \(x_5(t) = 24e^{5t}\). Next, \(x'_4 = 4x_4 + x_5 = 4x_4 + 24e^{5t}\). By undetermined coefficients, guess the particular solution \(x_{4p} = Ae^{5t}\) in which case \(x'_4 - 4x_4 = Ae^{5t}\), so \(A = 24\) and we have (using that \(x_4(0) = 30\)), \(x_4(t) = 6e^{4t} + 24e^{5t}\).

For \(x_3\), we have that \(x'_3 - 3x_3 = x_4 = 6e^{4t} + 24e^{5t}\). For \(x_{3p} = Be^{4t} + Ce^{5t}\) we have \(x'_3 - 3x_3 = Be^{4t} + 2Ce^{5t}\), so \(B = 6\) and \(C = 12\) and we have (using that \(x_3(0) = 20\)) \(x_3 = 2e^{3t} + 6e^{4t} + 12e^{5t}\).
Next, for $x_2$, we have that $x'_2 - 2x_2 = x_3 = 2e^{3t} + 6e^{4t} + 12e^{5t}$. For $x_{2p} = Ee^{3t} + Fe^{4t} + Ge^{5t}$ we have $x'_{2p} - 2x_{2p} = Ee^{3t} + 2Fe^{4t} + 3Ge^{5t}$. so $E = 2$, $F = 3$ and $G = 4$. Using $x_2(0) = 10$, we conclude that $x_2 = e^{2t} + 2e^{3t} + 3e^{4t} + 4e^{5t}$.

Finally, for $x_1$ we have that $x'_1 - x_1 = x_2 = e^{2t} + 2e^{3t} + 3e^{4t} + 4e^{5t}$. For $x_{1p} = He^{2t} + Je^{3t} + Ke^{4t} + Le^{5t}$ we have $x'_{1p} - x_{1p} = He^{2t} + 2Je^{3t} + 3Ke^{4t} + 4Le^{5t}$, so $H = 1$, $J = 1$, $K = 1$ and $L = 1$. Using $x_1(0) = 5$, we conclude that

$$x_1 = e^t + e^{2t} + e^{3t} + e^{4t} + e^{5t}.$$

4. Suppose $x' = Ax$, where

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $x$ satisfies the initial condition

$$x(0) = \begin{bmatrix} 1 \\ -2 \\ 8 \\ -18 \\ 24 \end{bmatrix}.$$ 

What is $x_1(t)$?

It might be easiest to do this one by “cleverness”. Note that the last equation says $x'_5 = x_5$ with $x_5(0) = 24$. Therefore $x_5(t) = 24e^t$. Next, $x'_4 = 2x_4 + x_5 = 2x_4 + 24e^t$. By undetermined coefficients, guess the particular solution $x_{4p} = Ae^t$ in which case $x'_{4p} - 2x_{4p} = -Ae^t$, so $A = -24$ and we have (using that $x_4(0) = -18$), $x_4(t) = 6e^{2t} - 24e^t$.

For $x_3$, we have that $x'_3 - 3x_3 = x_4 = 6e^{2t} - 24e^t$. For $x_{3p} = Be^{2t} + Ce^t$ we have $x'_{3p} - 3x_{3p} = -Be^{2t} - 2Ce^t$, so $B = -6$ and $C = 12$ and we have (using that $x_3(0) = 8$) $x_3 = 2e^{3t} - 6e^{2t} + 12e^t$.

Next, for $x_2$, we have that $x'_2 - 4x_2 = x_3 = 2e^{3t} - 6e^{2t} + 12e^t$. For $x_{2p} = Ee^{3t} + Fe^{2t} + Ge^t$ we have $x'_{2p} - 4x_{2p} = -Ee^{3t} - 2Fe^{2t} - 3Ge^t$. so $E = -2$, $F = 3$ and $G = -4$. Using $x_2(0) = -2$, we conclude that $x_2 = e^{4t} - 2e^{3t} + 3e^{2t} - 4e^t$.

Finally, for $x_1$ we have that $x'_1 - 5x_1 = x_2 = e^{4t} - 2e^{3t} + 3e^{2t} - 4e^t$. For $x_{1p} = He^{4t} + Je^{3t} + Ke^{2t} + Le^t$ we have $x'_{1p} - 5x_{1p} = -He^{4t} - 2Je^{3t} - 3Ke^{2t} - 4Le^t$, so $H = -1$, $J = 1$, $K = -1$ and $L = 1$. Using $x_1(0) = 1$, we conclude that

$$x_1 = e^t - e^{2t} + e^{3t} - e^{4t} + e^{5t}.$$
4. Suppose $\mathbf{x}' = A \mathbf{x}$, where

$$
\mathbf{x} = \begin{bmatrix}
  x_1(t) \\
  x_2(t) \\
  x_3(t) \\
  x_4(t) \\
  x_5(t)
\end{bmatrix}
\quad \text{and} \quad
A = \begin{bmatrix}
  5 & 1 & 0 & 0 & 0 \\
  0 & 4 & 1 & 0 & 0 \\
  0 & 0 & 3 & 1 & 0 \\
  0 & 0 & 0 & 2 & 1 \\
  0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

and $\mathbf{x}$ satisfies the initial condition $\mathbf{x}(0) = \begin{bmatrix}
  3 \\
  -6 \\
  14 \\
  -24 \\
  24
\end{bmatrix}$.

What is $x_1(t)$?

It might be easiest to do this one by "cleverness". Note that the last equation says $x'_5 = x_5$ with $x_5(0) = 24$. Therefore $x_5(t) = 24e^t$. Next, $x'_4 = 2x_4 + x_5 = 2x_4 + 24e^t$. By undetermined coefficients, guess the particular solution $x_{4p} = Ae^t$ in which case $x'_{4p} - 2x_{4p} = -Ae^t$, so $A = -24$ and we have (using that $x_4(0) = -24$), $x_4(t) = -24e^t$.

For $x_3$, we have that $x'_3 - 3x_3 = x_4 = -24e^t$. For $x_{3p} = Ce^t$ we have $x'_{3p} - 3x_{3p} = -2Ce^t$, so $C = 12$ and we have (using that $x_3(0) = 14$) $x_3 = 2e^{3t} + 12e^t$.

Next, for $x_2$, we have that $x'_2 - 4x_2 = x_3 = 2e^{3t} + 12e^t$. For $x_{2p} = Ee^{3t} + Ge^t$ we have $x'_{2p} - 4x_{2p} = -Ee^{3t} - 3Ge^t$. so $E = -2$ and $G = -4$. Using $x_2(0) = -6$, we conclude that $x_2 = -2e^{3t} - 4e^t$.

Finally, for $x_1$ we have that $x'_1 - 5x_1 = x_2 = -2e^{3t} - 4e^t$. For $x_{1p} = Je^{3t} + Le^t$ we have $x'_{1p} - 5x_{1p} = -2Je^{3t} - 4Le^t$, so $J = 1$ and $L = 1$. Using $x_1(0) = 3$, we conclude that

$$x_1 = e^t + e^{3t} + e^{5t}.$$
and \( \mathbf{x} \) satisfies the initial condition

\[
\mathbf{x}(0) = \begin{bmatrix} 15 \\ -20 \\ 30 \\ -36 \\ 24 \end{bmatrix}.
\]

What is \( x_1(t) \)?

This time, we’ll use the standard eigenvalue/eigenvector method. Since \( A \) is upper triangular, we know the eigenvalues are the diagonal entries, namely 1, 2, 3, 4 and 5. Since \( A \) has five distinct eigenvalues, we know that \( A \) is diagonalizable.

For \( \lambda = 1 \) we need the kernel of

\[
\begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

which is spanned by

\[
\begin{bmatrix} 1 \\ -4 \\ 12 \\ -24 \\ 24 \end{bmatrix}
\]

For \( \lambda = 2 \) we need the kernel of

\[
\begin{bmatrix} 3 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}
\]

which is spanned by

\[
\begin{bmatrix} 1 \\ -3 \\ 6 \\ -6 \\ 0 \end{bmatrix}
\]

For \( \lambda = 3 \) we need the kernel of

\[
\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}
\]

which is spanned by

\[
\begin{bmatrix} 1 \\ -2 \\ 2 \\ 0 \\ 0 \end{bmatrix}
\]

For \( \lambda = 4 \) we need the kernel of

\[
\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}
\]

which is spanned by

\[
\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]

And for \( \lambda = 5 \) we need the kernel of

\[
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}
\]

which is spanned by

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\]
Therefore, the general solution of $x' = Ax$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} + c_4 e^{4t} + c_5 e^{5t}.$$

Working backward from $x_5$, it is pretty easy to see that to satisfy the initial conditions we need $c_1 = 1, c_2 = 2, c_3 = 3, c_4 = 4$ and $c_5 = 5$. So

$$x_1(t) = e^t + 2e^{2t} + 3e^{3t} + 4e^{4t} + 5e^{5t}.$$

4. Suppose $x' = Ax$, where

$$x = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

and $x$ satisfies the initial condition

$$x(0) = \begin{bmatrix} 2 \\ 4 \\ 12 \\ 24 \\ 24 \end{bmatrix}.$$

What is $x_1(t)$?

It might be easiest to do this one by "cleverness". Note that the last equation says $x'_5 = 5x_5$ with $x_5(0) = 24$. Therefore $x_5(t) = 24 e^{5t}$. Next, $x'_4 = 4x_4 + x_5 = 4x_4 + 24 e^{5t}$. By undetermined coefficients, guess the particular solution $x_{4p} = Ae^{5t}$ in which case $x'_{4p} - 4x_{4p} = Ae^{5t}$, so $A = 24$ and we have (using that $x_4(0) = 24$), $x_4(t) = 24 e^{5t}$.

For $x_3$, we have that $x'_3 - 3x_3 = x_4 = 24 e^{5t}$. For $x_{3p} = Ce^{5t}$ we have $x'_{3p} - 3x_{3p} = 2C e^{5t}$, so $C = 12$ and we have (using that $x_3(0) = 12$) $x_3 = 12 e^{5t}$.

Next, for $x_2$, we have that $x'_2 - 2x_2 = x_3 = 12 e^{5t}$. For $x_{2p} = Ge^{5t}$ we have $x'_{2p} - 2x_{2p} = 3G e^{5t}$, so $G = 4$. Using $x_2(0) = 0$, we conclude that $x_2 = 4 e^{5t}$.

Finally, for $x_1$ we have that $x'_1 - x_1 = x_2 = 4 e^{5t}$. For $x_{1p} = Le^{5t}$ we have $x'_{1p} - x_{1p} = 4L e^{5t}$, so $L = 1$. Using $x_1(0) = 0$, we conclude that

$$x_1 = e^t + e^{5t}.$$
5. Find and classify all the critical points of the system

\[
\begin{align*}
\frac{dx}{dt} &= y^3 - y \\
\frac{dy}{dt} &= x^2 - x
\end{align*}
\]

Then draw (and label with arrows) several representative trajectories in the phase plane.

If \( y^3 - y = 0 \), then \( y \in \{-1, 0, 1\} \). And if \( x^2 - x = 0 \) then \( x \in \{0, 1\} \). So there are six critical points, \((0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1)\). The Jacobian matrix is

\[
J = \begin{bmatrix}
0 & 3y^2 - 1 \\
2x - 1 & 0
\end{bmatrix}
\]

At \((0, -1)\) and \((0, 1)\), we have \( J = \begin{bmatrix} 0 & 2 \\
-1 & 0 \end{bmatrix} \). The trace is zero and the determinant is positive (and the divergence of the original system is zero), so these are centers. A check of the sign of \( x' \) shows the curves go clockwise.

At \((0, 0)\) we have \( J = \begin{bmatrix} 0 & -1 \\
-1 & 0 \end{bmatrix} \). The determinant is negative so this is a saddle point, trajectories come in along \([1, 1]\) and go out along \([1, -1]\).

At \((1, -1)\) and \((1, 1)\) we have \( J = \begin{bmatrix} 0 & 2 \\
1 & 0 \end{bmatrix} \). The determinant is negative so these are saddle points. Trajectories come in along \([\sqrt{2}, -1]\) and go out along \([\sqrt{2}, 1]\).

Finally, at \((1, 0)\) we have \( J = \begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix} \). The determinant is positive and the trace is zero so this is a center, and the motion is counterclockwise.

Figure 21: Solutions of \( x' = y^3 - y, \ y' = x^2 - x \).
5. Find and classify all the critical points of the system

\[
\begin{align*}
\frac{dx}{dt} &= y^3 - y \\
\frac{dy}{dt} &= x - x^2
\end{align*}
\]

Then draw (and label with arrows) several representative trajectories in the phase plane.

If \( y^3 - y = 0 \), then \( y \in \{-1, 0, 1\} \). And if \( x - x^2 = 0 \) then \( x \in \{0, 1\} \). So there are six critical points, \((0, -1), (0, 0), (0, 1), (1, -1), (1, 0), (1, 1)\). The Jacobian matrix is

\[
J = \begin{bmatrix}
0 & 3y^2 - 1 \\
1 - 2x & 0
\end{bmatrix}
\]

At \((0, -1)\) and \((0, 1)\), we have \( J = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \). The trace is zero and the determinant is negative so these are saddle points. Trajectories go out along \([\sqrt{2}, 1]\) and come in along \([\sqrt{2}, -1]\).

At \((0, 0)\) we have \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). The determinant is positive and the trace is zero (and the divergence of the original system is zero), so this is a center, and a check of the sign of \( x' \) shows that the motion is counterclockwise.

At \((1, -1)\) and \((1, 1)\) we have \( J = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix} \). The determinant is positive and the trace is zero, so these are centers and the motion is clockwise.

Finally, at \((1, 0)\) we have \( J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \). The determinant is negative so this is a saddle point. Trajectories come in along \([1, 1]\) and go out along \([1, -1]\).

Figure 22: Solutions of \( x' = y^3 - y, \ y' = x - x^2 \).
5. Find and classify all the critical points of the system

\[
\begin{align*}
\frac{dx}{dt} &= y - y^2 \\
\frac{dy}{dt} &= x^3 - x
\end{align*}
\]

Then draw (and label with arrows) several representative trajectories in the phase plane.

If \(x^3 - x = 0\), then \(x \in \{-1, 0, 1\}\). And if \(y - y^2 = 0\) then \(y \in \{0, 1\}\). So there are six critical points, \((-1, 0), (0, 0), (1, 0), (-1, 1), (0, 1), (1, 1)\). The Jacobian matrix is

\[
J = \begin{bmatrix} 0 & 1 - 2y \\ 3x^2 - 1 & 0 \end{bmatrix}
\]

At \((-1, 0)\) and \((1, 0)\), we have \(J = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\). The trace is zero and the determinant is negative so these are saddle points. Trajectories go out along \([1, \sqrt{2}]\) and come in along \([-1, \sqrt{2}]\).

At \((0, 0)\) we have \(J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\). The determinant is positive and the trace is zero (and the divergence of the original system is zero), so this is a center, and a check of the sign of \(x'\) shows that the motion is clockwise.

At \((-1, 1)\) and \((1, 1)\) we have \(J = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}\). The determinant is positive and the trace is zero, so these are centers and the motion is counterclockwise.

Finally, at \((0, 1)\) we have \(J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\). The determinant is negative so this is a saddle point. Trajectories come in along \([1, 1]\) and go out along \([1, -1]\).

![Figure 23: Solutions of \(x' = y - y^2\), \(y' = x^3 - x\).](image)
5. Find and classify all the critical points of the system

\[
\frac{dx}{dt} = y^2 - y \\
\frac{dy}{dt} = x - x^3
\]

Then draw (and label with arrows) several representative trajectories in the phase plane.

If \( x - x^3 = 0 \), then \( x \in \{-1, 0, 1\} \). And if \( y^2 - y = 0 \) then \( y \in \{0, 1\} \). So there are six critical points, \((-1, 0), (0, 0), (1, 0), (-1, 1), (0, 1), (1, 1)\). The Jacobian matrix is

\[
J = \begin{bmatrix}
0 & 2y - 1 \\
1 - 3x^2 & 0
\end{bmatrix}
\]

At \((-1, 0)\) and \((1, 0)\), we have \( J = \begin{bmatrix} 0 & -1 \\ -2 & 0 \end{bmatrix} \). The trace is zero and the determinant is negative so these are saddle points. Trajectories come in along \([-1, \sqrt{2}]\) and go out along \([-1, \sqrt{2}]\).

At \((0, 0)\) we have \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). The determinant is positive and the trace is zero (and the divergence of the original system is zero), so this is a center, and a check of the sign of \( x' \) shows that the motion is counterclockwise.

At \((-1, 1)\) and \((1, 1)\) we have \( J = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \). The determinant is positive and the trace is zero, so these are centers and the motion is clockwise.

Finally, at \((0, 1)\) we have \( J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). The determinant is negative so this is a saddle point. Trajectories go out along \([1, 1]\) and come in along \([1, -1]\).

Figure 24: Solutions of \( x' = y^2 - y, \ y' = x - x^3 \).
5. Find and classify all the critical points of the system

\[
\begin{align*}
\frac{dx}{dt} &= y^2 - y \\
\frac{dy}{dt} &= x^2 - x
\end{align*}
\]

Then draw (and label with arrows) several representative trajectories in the phase plane.

If \( y^2 - y = 0 \), then \( y \in \{0, 1\} \). And if \( x^2 - x = 0 \) then \( x \in \{0, 1\} \). So there are four critical points, \((0, 0), (0, 1), (1, 0), (1, 1)\). The Jacobian matrix is

\[
J = \begin{bmatrix}
0 & 2y - 1 \\
2x - 1 & 0
\end{bmatrix}
\]

At \((0, 1)\), we have \( J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \). The trace is zero and the determinant is positive (and the divergence of the original system is zero), so this is a center. A check of the sign of \( x' \) shows the curves go clockwise.

At \((0, 0)\) we have \( J = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \). The determinant is negative so this is a saddle point. Trajectories come in along \([0, 1]\) and go out along \([1, -1]\).

At \((1, 1)\) we have \( J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \). The determinant is negative so this is a saddle point. Trajectories come in along \([1, -1]\) and go out along \([1, 1]\).

Finally, at \((1, 0)\) we have \( J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \). The determinant is positive and the trace is zero so this is a center, and the motion is counterclockwise.

**Figure 25:** Solutions of \( x' = y^2 - y, \ y' = x^2 - x \).