MATH 240 – Practice problems for First Midterm Exam - Spring 2015

1. The determinant of the matrix
\[
\begin{pmatrix}
2 & 0 & 1 & 0 \\
2 & 3 & 3 & 1 \\
-3 & 2 & 1 & 2 \\
0 & 1 & 2 & 1 \\
\end{pmatrix}
\]
is $-6$. What is the determinant of the matrix
\[
\begin{pmatrix}
-3 & 2 & 1 & 2 \\
4 & 6 & 6 & 2 \\
2 & 0 & 1 & 0 \\
0 & 1 & 2 & 1 \\
\end{pmatrix}
\]

of the matrix
\[
\begin{pmatrix}
4 & 3 & 4 & 1 \\
2 & 3 & 3 & 1 \\
0 & 1 & 2 & 1 \\
-3 & 2 & 1 & 2 \\
\end{pmatrix}
\]

of the matrix
\[
\begin{pmatrix}
-2 & -3 & -3 & -1 \\
3 & -2 & -1 & -2 \\
0 & -1 & -2 & -1 \\
-2 & 0 & -1 & 0 \\
\end{pmatrix}
\]

of the matrix
\[
\begin{pmatrix}
4 & 0 & 2 & 0 \\
0 & 1 & 2 & 1 \\
2 & 3 & 3 & 1 \\
-3 & 2 & 1 & 2 \\
\end{pmatrix}
\]

2. For each of the following subsets of the vector space $P_4$ of polynomials of degree less than or equal to 4, say whether or not it is a (vector) subspace of $P_4$. If it is not a subspace, explain why not. If it is a subspace, give its dimension and a basis for the subspace.

(a) The set of polynomials in $P_4$ that are even functions (i.e., for which $p(-x) = p(x)$).

(b) The set of polynomials in $P_4$ that are odd functions (i.e., for which $p(-x) = -p(x)$).

(c) The set of polynomials in $P_4$ that satisfy $p(0) = 1$ and $p(1) = 2$.

(d) The set of polynomials in $P_4$ that satisfy $p(0) = 0$ and $p(1) = 0$.

(e) The set of polynomials in $P_4$ that satisfy $p(1) = 0, p'(1) = 0$ and $p''(1) = 0$.

(f) The set of polynomials in $P_4$ that satisfy $p(1) = 1$ and $p'(1) = 2$. 
3. Consider the matrix $A(k) = \begin{bmatrix} 1 & 1 & -2 \\ 1 & k & 0 \\ -1 & 2 & k \end{bmatrix}$.

(a) There are two values of $k$ for which the rank of the matrix $A(k)$ is less than three. What are they?

(b) For each of those values of $k$, find a basis for the nullspace of $A(k)$.

(c) For one of the values of $k$, it is possible to solve $A(k)x = b$, where

$$b = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}.$$ 

What is the general solution of this problem for this value of $k$?

4. For the matrix

$$M = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

determine the dimension of the subspace of 3-by-3 matrices $X$ for which

$$MX = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Also give a basis for this subspace.

5. Let $S = \{1, x, x^2\}$ be the standard basis for the vector space $P_2$ of polynomials of degree less than or equal to 2.

(a) Show that $B = \{1 + x, 1 + x^2, x + x^2\}$ is another basis for $P_2$.

(b) What is the change-of-basis matrix $P_{S\rightarrow B}$ (in other words how do you go from expressing a polynomial as $a_1(1 + x) + a_2(1 + x^2) + a_3(x + x^2)$ to expressing it as $b_1(1) + b_2(x) + b_3(x^2)$)?

(c) What is the change-of-basis matrix $P_{B\rightarrow S}$?

(d) What is the matrix that represents the linear mapping that sends $p(x)$ to $p'(x) + 2p(x)$ with respect to the standard basis $S$?

(e) What is the matrix that represents the linear mapping in part (d) with respect to the basis $B$?
6. (a) Can the vector \[
\begin{bmatrix}
2 \\
1 \\
5 \\
0
\end{bmatrix}
\] be represented as a linear combination of the vectors \[
\begin{bmatrix}
1 \\
0 \\
2 \\
1
\end{bmatrix}, \quad \begin{bmatrix}
1 \\
1 \\
6 \\
2
\end{bmatrix}, \quad \begin{bmatrix}
0 \\
0 \\
1 \\
1
\end{bmatrix}
\] ? If not, explain why not. If so, how? (Be precise – if there is more than one way to do it, give all possible ways).

(b) Same question, but for the vector \[
\begin{bmatrix}
2 \\
1 \\
4 \\
1
\end{bmatrix}
\]

7. Let \( A = \begin{bmatrix}
1 & 0 & -1 & -2 & 0 & 0 \\
-2 & -1 & 0 & 2 & 0 & -1
\end{bmatrix} \).

(a) Explain why the matrix \( A^T(AA^T)^{-1} \) would be a right inverse for \( A \), provided it exists.

(b) Calculate \( A^T(AA^T)^{-1} \) and show that it is a right inverse for \( A \). (Sorry about the fractions!)