MATH 240 – Practice problems for Second Midterm Exam - Spring 2015

1. For the matrix \( A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \):

(a) Find a diagonal matrix \( D \) and an invertible matrix \( P \) such that \( PDP^{-1} = A \).

(b) Calculate \( \lim_{n \to \infty} A^n \).

(c) Calculate \( e^{tA} \)

(a) Since \( A \) has two identical rows, we know \( \det(A) = 0 \), so \( \lambda = 0 \) is an eigenvalue and has eigenvector \( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \). So the other eigenvalue must be \( \lambda = 1 \) (since the trace of the matrix is 1) and for \( \lambda = 1 \) it is easy to see that the eigenvector is \( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \). So we can set \( D = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \) and \( P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \). It’s also easy to calculate that \( \det P = 2 \) and so \( P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \).

(b) Since it’s clear that \( D^n = D \), we have \( A^n = PD^nP^{-1} = PDP^{-1} = A \). So \( \lim_{n \to \infty} A^n = A \).

(c) Since \( e^{tD} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \), we have

\[
 e^{tA} = Pe^{tD}P^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
 = \begin{bmatrix} e^t & -1 \\ e^t & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t + \frac{1}{2} & \frac{1}{2}e^t - \frac{1}{2} \\ \frac{1}{2}e^t - \frac{1}{2} & \frac{1}{2}e^t + \frac{1}{2} \end{bmatrix}
\]

2. Consider the linear differential operator \( L : y \mapsto y''' + 4y'' + 4y' \). Let \( K \) be the kernel of \( L \), i.e., the set of solutions of \( y''' + 4y'' + 4y' = 0 \).

(a) Give a basis for \( K \). What is the dimension of \( K \)?

(b) Let \( D \) be the usual derivative operator that sends a function \( y \) to its derivative \( y' \). Explain why \( D : K \to K \) (in other words why the derivative of any solution of \( y''' + 4y'' + 4y' = 0 \) is also a solution of the same equation.

(c) Since \( D : K \to K \), and \( K \) is finite-dimensional, you can write the matrix of \( D \) with respect to a basis of \( K \). Do this for the basis you gave as the answer to part (a).

(d) What are the eigenvalues, eigenvectors and determinant of the matrix of part (c)? To what functions do the eigenvectors correspond?

(e) Is the matrix of part (c) diagonalizable? Why or why not?
(a) To solve \( y'' + 4y'' + 4y' = 0 \), we need to find the roots of the polynomial \( r^3 + 4r^2 + 4r = 0 \), i.e., of \( r(r + 2)^2 = 0 \). The solutions are \( r = 0, -2, -2 \), so the general solution of the differential equation is \( y = c_1 + c_2e^{-2t} + c_3te^{-2t} \). A basis of \( K \) is thus \( \{ 1, e^{-2t}, te^{-2t} \} \) and the dimension of \( K \) is three.

(b) It is true in general that the derivative of a solution of a linear differential equation with constant coefficients is also a solution. To see it in this case, suppose the function \( y_0 \) satisfies \( y'' + 4y'' + 4y' = 0 \). If we put \( y_0' \) into the left side of the equation we get:

\[
(y_0'')'' + 4(y_0'')' + 4(y_0')' = y_0'' + 4y_0'' + 4y_0' = (y_0'' + 4y_0'' + 4y_0')'
\]

which is zero because the expression in the parentheses is zero by assumption. Therefore \( D \) maps \( K \) to \( K \).

(c) We calculate \( D(1) = 0, D(e^{-2t}) = -2e^{-2t} \) and \( D(te^{-2t}) = e^{-2t} - 2te^{-2t} \). So the matrix of \( D \) with respect to this basis is:

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & -2 & 1 \\
0 & 0 & -2
\end{bmatrix}
\]

(d) Since the matrix is upper triangular, the eigenvalues are on the diagonal — they are 0, \(-2\) and \(-2\), and the determinant of the matrix is 4. The eigenvectors are

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\text{ for } \lambda = 0, \quad \text{and} \quad 
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}
\text{ for } \lambda = -2.
\]

The \( \lambda = 0 \) eigenvector corresponds to the constant function \( y(t) = 1 \), and the \( \lambda = -2 \) eigenvector corresponds to \( y(t) = e^{-2t} \).

(c) Since \( \lambda = -2 \) has only one linearly independent eigenvector, the matrix in part (c) is not diagonalizable.

3. Solve the initial-value problem for:

(a) the differential equation \( y'' + 5y' + 6y = 24e^t \) together with the initial conditions \( y(0) = 0 \) and \( y'(0) = 0 \).

(b) the differential equation \( y'' + 5y' + 6y = 3e^{-2t} \) together with the initial conditions \( y(0) = 0 \), \( y'(0) = 1 \).

(c) the differential equation \( y'' + 2y' + 9y = t \) together with the initial conditions \( y(0) = 1 \), \( y'(0) = 0 \).

(d) the differential equation \( y'' + 2ky' + 4y = e^{-t} \) together with the initial conditions \( y(0) = 0 \), \( y'(0) = 0 \). (Be careful to indicate different the behavior of the solutions for different values of \( k \).) Draw sketches of the graphs of the solutions for enough different values of \( k \) to get a sense of how “tuning” the value of \( k \) affects the nature of the solution.

(a) The auxiliary polynomial is \( r^2 + 5r + 6 = (r + 3)(r + 2) \) so the solution of the corresponding homogeneous equation is \( y_c = c_1e^{-3t} + c_2e^{-2t} \). For a particular solution, we can guess \( y_p = Ae^t \).
and see that \( y'' + 5y' + 6y = 12Ae^t \), so we need \( A = 2 \). So the general solution of the equation is
\[
y(t) = 2e^t + c_1e^{-3t} + c_2e^{-2t}.
\]
To get \( y(0) = 0 \) we need \( 2 + c_1 + c_2 = 0 \) and to get \( y'(0) = 0 \) we need \( 2 - 3c_1 - 2c_2 = 0 \). Solve and get \( c_1 = 6 \) and \( c_2 = -8 \). Therefore the solution of the initial-value problem is
\[
y(t) = 2e^t + 6e^{-3t} - 8e^{-2t}.
\]

(b) The homogeneous equation is the same as in (a), but since the right side is in the kernel of the left side, we need to guess \( y_p = Ae^{-2t} \). This gives \( y'' + 5y' + 6y = Ae^{-2t} \), and so we need \( A = 3 \). The general solution of the differential equation is:
\[
y(t) = 3te^{-2t} + c_1e^{-3t} + c_2e^{-2t}.
\]
To get \( y(0) = 0 \) we need \( c_1 + c_2 = 0 \) and to get \( y'(0) = 1 \) we need \( 3 - 3c_1 - 2c_2 = 1 \). Solve and get \( c_1 = 2 \) and \( c_2 = -2 \). Therefore the solution of the initial-value problem is
\[
y(t) = 3te^{-2t} + 2e^{-3t} - 2e^{-2t}.
\]

(c) The auxiliary polynomial this time is \( r^2 + 2r + 9 \), whose roots (by the quadratic formula) are \(-1 \pm i2\sqrt{2}\). So the solution of the corresponding homogeneous equation is \( y_c = c_1e^{-t}\cos 2\sqrt{2}t + c_2e^{-t}\sin 2\sqrt{2}t \). For a particular solution, we guess \( y_p = A + Bt \), and see that \( y'' + 2y' + 9y + p = (9A + 2B) + 9Bt \), and this is equal to \( t \) if \( B = \frac{1}{9} \) and \( A = -\frac{2}{81} \). So the general solution of the equation is
\[
y(t) = -\frac{2}{81} + \frac{1}{9}t + c_1e^{-t}\cos 2\sqrt{2}t + c_2e^{-t}\sin 2\sqrt{2}t.
\]
To get \( y(0) = 1 \), we need \(-\frac{2}{81} + c_1 = 1 \) which implies \( c_1 = \frac{83}{81} \), and then \( y'(0) = \frac{1}{9} - c_1 + 2\sqrt{2}c_2 = -\frac{4}{81} + 2\sqrt{2}c_2 \). This will be zero if \( c_2 = -\frac{37\sqrt{2}}{162} \). Therefore the solution of the initial-value problem is
\[
y(t) = -\frac{2}{81} + \frac{1}{9}t + \frac{83}{81}e^{-t}\cos 2\sqrt{2}t + \frac{37\sqrt{2}}{162} e^{-t}\sin 2\sqrt{2}t.
\]

(d) The auxiliary polynomial this time is \( r^2 + 2kr + 4 \), whose roots (by the quadratic formula) are \( r = -k \pm \sqrt{k^2 - 4} \). To begin our analysis, we’ll write the solution of the problem assuming that the roots of the auxiliary polynomial are simple (i.e., that \( k \neq \pm 2 \)) and that neither one of them is \(-1 \) (this happens only for \( k = \frac{5}{2} \), as we’ll explain below). As long as \( k \) is neither of those numbers, we have that the solution of the corresponding homogeneous equation is \( y_c = c_1e^{-k+\sqrt{k^2-4})t} + c_2e^{-k-\sqrt{k^2-4})t} \). For a particular solution (as long as \( k \neq \frac{5}{2} \)), we guess \( y_p = Ae^{-t} \), and see that \( y'' + 2ky' + 4y = (1 - 2k + 4)e^{-t} = (5 - 2k)e^{-t} \) (this shows why we can’t have \( k = \frac{5}{2} \)). This will be equal to \( e^{-t} \) if \( A = 1/(5 - 2k) \). So the general solution of the equation is
\[
y(t) = \frac{1}{5 - 2k} e^{-t} + c_1e^{(-k+\sqrt{k^2-4})t} + c_2e^{(-k-\sqrt{k^2-4})t}.
\]
To get \( y(0) = 0 \) and \( y'(0) = 0 \) we need
\[
c_1 + c_2 = \frac{1}{2k - 5} \quad \text{and} \quad (-k + \sqrt{k^2 - 4})c_1 + (-k - \sqrt{k^2 - 4})c_2 = \frac{1}{5 - 2k}
\]
respectively. Multiply the first of these equations by \((-k + \sqrt{k^2 - 4})\) and subtract the second equation from the result to get

\[
2\sqrt{k^2 - 4}c_2 = \frac{-k + \sqrt{k^2 - 4} + 1}{2k - 5} \quad \text{or} \quad c_2 = \frac{-k + 1 + \sqrt{k^2 - 4}}{2\sqrt{k^2 - 4}(2k - 5)}.
\]

Likewise, multiply the first equation by \((-k - \sqrt{k^2 - 4})\) and subtract the second equation to get

\[
-2\sqrt{k^2 - 4}c_1 = \frac{-k - \sqrt{k^2 - 4} + 1}{2k - 5} \quad \text{or} \quad c_1 = \frac{k - 1 + \sqrt{k^2 - 4}}{2\sqrt{k^2 - 4}(2k - 5)}.
\]

So, unless \(k \notin \{-2, 2, \frac{3}{2}\}\), the solution of the initial-value problem is

\[
y(t) = \frac{1}{5 - 2k}e^{-t} + \frac{k - 1 + \sqrt{k^2 - 4}}{2\sqrt{k^2 - 4}(2k - 5)}e^{(-k + \sqrt{k^2 - 4})t} + \frac{-k + 1 + \sqrt{k^2 - 4}}{2\sqrt{k^2 - 4}(2k - 5)}e^{(-k - \sqrt{k^2 - 4})t}.
\]

Before we proceed to analyze this solution further, we find the solutions for the exceptional values of \(k\). If \(k = -2\), then the auxiliary equation is \((r - 2)^2 = 0\), so the solution of the homogeneous equation is \(y_c = c_1e^{2t} + c_2te^{2t}\). And \(1/(5 - 2k) = 1/9\), so the general solution of the differential equation is

\[
y(t) = \frac{1}{9}e^{-t} + c_1e^{2t} + c_2te^{2t}.
\]

For \(y(0) = 0\), we need \(c_1 = -1/9\), and then for \(y'(0) = 0\) we need \(c_2 = 1/3\). So the solution for \(k = -2\) is

\[
y(t) = \frac{1}{9}e^{-t} - \frac{1}{9}e^{2t} + \frac{1}{3}te^{2t}.
\]

Likewise, if \(k = 2\), then the auxiliary equation is \((r + 2)^2 = 0\), so the solution of the homogeneous equation is \(y_c = c_1e^{-2t} + c_2te^{-2t}\). And \(1/(5 - 2k) = 1\), so the general solution of the differential equation is

\[
y(t) = e^{-t} + c_1e^{-2t} + c_2te^{-2t}.
\]

For \(y(0) = 0\), we need \(c_1 = -1\) and then for \(y'(0) = 0\) we need \(c_2 = -1\) as well, so the solution for \(k = 2\) is

\[
y(t) = e^{-t} - e^{-2t} - te^{-2t}.
\]

Finally, if \(k = 5/2\), then the auxiliary equation is \(r^2 + 5r + 4 = 0\), i.e., \((r + 4)(r + 1) = 0\), so \(-1\) is a root, and we have to guess \(y_p = Ate^{-t}\). Then \(y''_p + 5y'_p + 4y_p = -2Ae^{-t} + 5Ae^{-t} = 3Ae^{-t}\), which equals \(e^{-t}\) if \(A = \frac{1}{3}\). So the general solution in this case is

\[
y(t) = \frac{1}{3}te^{-t} + c_1e^{-t} + c_2e^{-4t}.
\]

For \(y(0) = 0\), we need \(c_1 + c_2 = 0\) and for \(y'(0) = 0\) we need \(\frac{1}{3} - c_1 - 4c_2 = 0\). The solution of these equations is \(c_1 = -\frac{1}{9}\) and \(c_2 = \frac{1}{9}\) so the solution for \(k = \frac{5}{2}\) is

\[
y(t) = \frac{1}{3}te^{-t} - \frac{1}{9}e^{-t} + \frac{1}{9}e^{-4t}.
\]

Now let’s think about what the graphs of the solutions should look like for various values of \(k\). If \(k < -2\), then the coefficient of the \(e^{-t}\) term is positive, and the two roots of the auxiliary equation are real and positive. The coefficient of the exponential corresponding to the larger root (the larger
root will be \(-k + \sqrt{k^2 - 4}\) will also be positive. Therefore the graph of the solution will go toward \(\infty\) as \(t \to \pm \infty\) and will do it faster as \(t \to +\infty\). When \(0 < k < 2\), then the solution will decay to zero (still oscillating) as \(t \to \infty\), and will oscillate more and more violently for \(t < 0\). Finally, for \(t > 2\), the solution essentially looks like the graph of \(e^{-2t}\) times a positive constant. The “oddball” solutions for \(k = -2, 2, 0, 5/2\) represent transitions from one phase to the next.

Here are some pictures:

These are graphs of the solution for various values of \(k\) less than \(-2\) (actually, that stray line is for \(k = -1.99\)). They all exhibit pretty much the same behavior:

Next are some graphs for \(-2 < k < 0\). The oscillations grow less fast as \(k \to 0\).
Here is $k = 0$, as well as some values for $0 < k < 2$. Note that the oscillations are decaying as $t$ grows.

Finally, for $k > 2$, the solutions decay exponentially as $t \to \infty$.

For $-2 < t < 2$, the graph will oscillate because the $k^2 - 4$ under the square root will be negative, so there are actually sines and cosines multiplying $e^{-kt}$. If $k$ is between $-2$ and $0$ (i.e., negative), the graph will still grow exponentially as $t \to -\infty$ because of the $e^{-t}$ term, and will grow as $t \to \infty$ as well.
4. Write down a homogeneous linear second-order differential equation with constant coefficients, together with initial conditions at $t = 0$ so that the solution $y(t)$ will have the following properties:

- $y(0) = 0$
- $y(t)$ is increasing for all $t > 0$
- $\lim_{t \to \infty} y(t) = 6$.

(There are many possible answers – think about what function you might want the solution to be).

A function that has the specified properties is $y = 6 - 6e^{-t}$. And this function is a solution of the differential equation $y'' + y' = 0$. Clearly one initial condition is $y(0) = 0$, and we obtain the other by differentiating: $y' = 6e^{-t}$, so we have $y'(0) = 6$.

5. Find the general solution of the differential equation:

(a) $y'' + 4y' + 4y = t^{5/2}e^{-2t}$

(b) $(1 + t^2)y'' - 2ty' + 2y = 12$ (note: $y_1(t) = t$ is one solution of the associated homogeneous equation).

(a) The auxiliary equation is $r^2 + 4r + 4 = 0$, whose roots are $r = -2, -2$, and so the general solution of the homogeneous equation is $y_h = c_1e^{-2t} + c_2te^{-2t}$. We seek a particular solution in the form $y_p = y_1v_1 + y_2v_2$, where $y_1 = e^{-2t}$ and $y_2 = te^{-2t}$ and $v_1$ and $v_2$ are unknown functions. The variation of parameters formula tells us that $v_1' = -\frac{y_2}{W}(t^{5/2}e^{-2t})/W$ and $v_2' = y_1(t^{5/2}e^{-2t})/W$, where $W = y_1y_2' - y_2y_1'$ is the Wronskian of the two homogeneous solutions. So first we calculate: $W = e^{-2t}(e^{-2t} - 2te^{-2t}) - (2e^{-2t})(te^{-2t}) = e^{-4t}$, and then we have

$$v_1' = \frac{-((te^{-2t})(t^{5/2}e^{-2t}))}{e^{-4t}} = -t^{7/2}$$

so $v_1 = \frac{-2}{9}t^{9/2}$

and

$$v_2' = \frac{(e^{-2t})(t^{5/2}e^{-2t})}{e^{-4t}} = t^{5/2}$$

so $v_2 = \frac{2}{7}t^{7/2}$.

So we get our particular solution: $y_p = -\frac{2}{9}t^{9/2}(e^{-2t}) + \frac{2}{7}t^{7/2}(te^{-2t}) = \frac{4}{63}t^{9/2}e^{-2t}$. Thus, the general solution of the equation is

$$y(t) = \frac{4}{63}t^{9/2}e^{-2t} + c_1e^{-2t} + c_2te^{-2t}$$

(b) We use the method of “reduction of order”, that is, we assume the solution of the equation is $y(t) = tv(t)$ for some function $v(t)$, because $t$ is a solution of the homogeneous equation. We have $y' = tv' + v$ and $y'' = v'' + 2v'$, and substituting these into the equation we get:

$$(t + t^3)v'' + 2tv' = 12,$$

which is a linear first-order equation with $v'$ as the unknown function. We put it in the standard form by dividing by $t + t^3 = t(1 + t^2)$ and get

$$v'' + \frac{2}{t(1 + t^2)}v' = \frac{12}{t(1 + t^2)}$$
For the integrating factor, we’re going to need to integrate:
\[
\int \frac{2}{t(1 + t^2)} \, dt = \int \frac{2}{t} - \frac{2t}{1 + t^2} \, dt = 2 \ln t - \ln(1 + t^2).
\]
The integrating factor is the exponential of this, so we multiply both sides of the equation by 
\[
t^2/(1 + t^2)
\]
and obtain:
\[
\left( \frac{t^2}{1 + t^2} \, v' \right)' = \frac{12t}{(1 + t^2)^2}.
\]
Then integrate both sides to get:
\[
\frac{t^2}{1 + t^2} \, v' = -\frac{6}{1 + t^2} + C
\]
Next, isolate the \(v'\):
\[
v' = -\frac{6}{t^2} + C \left(1 + \frac{1}{t^2}\right)
\]
and integrate again to get:
\[
v = \frac{6}{t} + Ct - \frac{C}{t} + K
\]
so that
\[
y = tv = 6 + C(t^2 - 1) + Kt.
\]

6. The matrix \(A = \begin{bmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{bmatrix}\) is diagonalizable. Find a diagonal matrix \(D\) and an invertible matrix \(P\) such that \(PDP^{-1} = A\), and calculate \(e^{tA}\).

The characteristic polynomial of \(A\) is \(p(\lambda) = (1 - \lambda)(2 - \lambda)(-1 - \lambda) + 2 + 12 + (1 - \lambda) + 3(-1 - \lambda) - 8(2 - \lambda) = -\lambda^3 + 2\lambda^2 + 5\lambda - 6\). It is easy to see the \(\lambda = 1\) is a root of this polynomial, and then by synthetic (or long) division we see that \(p(\lambda) = -(\lambda - 1)(\lambda^2 - \lambda - 6) = -(\lambda - 1)(\lambda - 3)(\lambda + 2)\). So the eigenvalues of \(A\) are 1, 3, 2 (and \(A\) is diagonalizable since it has three different eigenvalues).

To find the eigenvector for \(\lambda = 1\) we row-reduce:
\[
\begin{bmatrix} 0 & -1 & 4 \\ 3 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2, \ R_1 \rightarrow R_1 - R_3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 1 & -4 \end{bmatrix}
\]
which gives us that the eigenvector corresponding to \(\lambda = 1\) is \([-1, 4, 1]^T\).

To find the eigenvector for \(\lambda = 3\), we row-reduce:
\[
\begin{bmatrix} -2 & -1 & 4 \\ 3 & -1 & -1 \\ 2 & 1 & -4 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2, \ R_1 \rightarrow R_1 + R_2} \begin{bmatrix} 1 & -2 & 3 \\ 0 & -5 & 10 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_2 \rightarrow 1/5 R_2, \ R_1 \rightarrow R_1 + 2R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}
\]
which gives us that the eigenvector corresponding to \(\lambda = 3\) is \([1, 2, 1]^T\).
Finally, to find the eigenvector for $\lambda = -2$ we row-reduce:

\[
\begin{bmatrix}
3 & -1 & 4 \\
3 & 4 & -1 \\
2 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 3 \\
0 & 10 & -10 \\
0 & 5 & -5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

which gives us that the eigenvector corresponding to $\lambda = -2$ is $[-1, 1, 1]^T$.

We put all this together and conclude that matrices $D$ and $P$ so that $A = PDP^{-1}$ are:

\[
D = \begin{bmatrix}
1 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & -2
\end{bmatrix}
\quad \text{and} \quad
P = \begin{bmatrix}
-1 & 1 & -1 \\
4 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

To calculate $e^{tA}$ we will need $P^{-1}$. So we row-reduce:

\[
\begin{bmatrix}
-1 & 1 & -1 \\
4 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -\frac{1}{2} & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 1 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & -\frac{1}{2} & \frac{1}{2} \\
0 & \frac{1}{2} & 0 \\
0 & 1 & -3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Therefore

\[
P^{-1} = \begin{bmatrix}
-\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
-\frac{1}{3} & -\frac{1}{3} & 1
\end{bmatrix}
\]

and so

\[
e^{tA} = P e^{tD} P^{-1} = \begin{bmatrix}
-1 & 1 & -1 \\
4 & 2 & 1 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
e^t & 0 & 0 \\
0 & e^{3t} & 0 \\
0 & 0 & e^{-2t}
\end{bmatrix}
\begin{bmatrix}
-\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
-\frac{1}{3} & -\frac{1}{3} & 1
\end{bmatrix}
\]

\[
= \frac{1}{6}
\begin{bmatrix}
-e^t & e^{3t} & -e^{-2t} \\
4e^t & 2e^{3t} & e^{-2t} \\
e^t & e^{3t} & e^{-2t}
\end{bmatrix}
\begin{bmatrix}
-1 & 2 & -3 \\
3 & 0 & 3 \\
-2 & -2 & 6
\end{bmatrix}
\]

\[
= \frac{1}{6}
\begin{bmatrix}
e^t + 3e^{3t} + 2e^{-2t} & -2e^t + 2e^{-2t} & 3e^t + 3e^{3t} - 6e^{-2t} \\
-4e^t + 6e^{3t} - 2e^{-2t} & 8e^t - 2e^{-2t} & -12e^t + 6e^{3t} - 6e^{-2t} \\
-e^t + 3e^{3t} - 2e^{-2t} & 2e^t - 2e^{-2t} & -3e^t + 3e^{3t} + 6e^{-2t}
\end{bmatrix}
\]