We’ve been using Fourier series to solve the heat and Laplace equations, but we haven’t paid much attention to the following questions:

- Do Fourier series converge?
- What do they converge to? Do they converge to the functions we expect them to?
- Can we differentiate Fourier series and obtain the expected answers (so that they really do represent solutions to PDEs)?
Fourier series "flavors"

We’ve seen several types of Fourier series:

- "Full" Fourier series (in solutions of Laplace equation on the disk):
  \[
  f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{2n\pi x}{L} \right) + b_n \sin \left( \frac{2n\pi x}{L} \right)
  \]

- Fourier sine series (zero boundary conditions on both ends)
  \[
  f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{L} \right)
  \]

- Fourier cosine series (zero derivative on both ends, i.e., insulated ends)
  \[
  f(x) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{n\pi x}{L} \right)
  \]
We’ve also seen ”mixed” flavors, with $f = 0$ at one end and $f' = 0$ at the other:

- $f(x) = \sum_{n=0}^{\infty} a_n \cos \left( \frac{(2n + 1)\pi x}{2L} \right)$ has $f'(0) = 0, f(L) = 0$

- $f(x) = \sum_{n=0}^{\infty} b_n \sin \left( \frac{(2n + 1)\pi x}{2L} \right)$ has $f(0) = 0, f'(L) = 0$

One thing we know is that whatever these Fourier series converge to will be a periodic function, with period $L$ in the full Fourier case, period $2L$ in the sine series and cosine series cases, and period $4L$ in the mixed case.
Even and odd

We know a little more than that: since $\sin x$ is an odd function and $\cos x$ is an even function, the Fourier sine series will produce an odd function and the Fourier cosine series an even one.

So, if you start out with the function $f(x) = x$ on the interval $0 \leq x \leq 2$:
Full Fourier series

The "full" Fourier series for \( f(x) = x \) on \( 0 \leq x \leq 2 \) is

\[
1 + \sum_{n=1}^{\infty} \frac{-2}{n\pi} \sin \left( \frac{n\pi x}{2} \right)
\]

which gives the periodic extension of \( f \):
The Fourier sine series for $f(x) = x$ on $0 \leq x \leq 2$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}4}{n\pi} \sin \left( \frac{n\pi x}{2} \right)$$

which gives the odd periodic extension of $f$. 

\[ <\text{Graph showing the odd periodic extension of } f(x) = x\text{, } 0 \leq x \leq 2 >> \]
Likewise, the Fourier cosine series for \( f(x) = x \) on \( 0 \leq x \leq 2 \) is

\[
1 + \sum_{n=0}^{\infty} \frac{-8}{(2n + 1)^2 \pi^2} \cos \left( \frac{n\pi x}{2} \right)
\]

which gives the even periodic extension of \( f \):

![Graph of function](image-url)
Observations

- The sum of the cosine series is continuous (but its derivative is not), and its coefficients go to zero like $1/n^2$, whereas the full Fourier series and the sine series are not continuous and their coefficients go to zero like $1/n$ (slower) — this is not a coincidence.
- At $x = \pm 2, \pm 4, \ldots$ the sine series converges to 0. This is not a coincidence either.
- To what do the mixed Fourier series converge?
Inner product spaces

To understand more about the convergence of Fourier series, we will discuss inner-product spaces in general. These are vector spaces (we know what they are) that are equipped with an inner product — that is, a "positive-definite symmetric bilinear function".

- **Bilinear means:** \( \langle \alpha v + \beta w, x \rangle = \alpha \langle v, x \rangle + \beta \langle w, x \rangle \) and \( \langle v, \alpha w + \beta x \rangle = \alpha \langle v, w \rangle + \beta \langle v, x \rangle \) for all vectors \( v, w, x \) and scalars \( \alpha, \beta \).

- **Symmetric means:** \( \langle v, w \rangle = \langle w, v \rangle \)

- **Positive definite means:** \( \langle v, v \rangle \geq 0 \) for all \( v \), and \( \langle v, v \rangle = 0 \) if and only if \( v \) is the zero vector.
The usual dot product of vectors in $\mathbb{R}^3$ is the basic example of an inner product. And just as in that case, we use the inner product to do geometry:

- We define the length of a vector to be $|v| = \sqrt{\langle v, v \rangle}$ (and the distance between two vectors, if we want to consider them as points, is $|v - w|$).
- Then we have the *Schwarz inequality* $|\langle v, w \rangle| \leq |v||w|$ and the *triangle inequality* $|v \pm w| \leq |v| + |w|$.
- We can define the angle between two vectors via $\cos \theta = \frac{\langle v, w \rangle}{|v||w|}$. This makes sense because of the Schwarz inequality.
We will think of functions as vectors. Typically, we’ll take the set of functions defined on a specific interval (like $0 \leq x \leq L$) and satisfying some analytic condition (i.e., continuous, differentiable, integrable, piecewise smooth). We can add such functions together and multiply them by scalars, so we have a vector space. For our inner product, we’ll define:

$$\langle f, g \rangle = \int_0^L f(x)g(x) \, dx.$$ 

It’s easy to see that this is bilinear and symmetric. It’s also positive definite if we don’t allow the functions to be too weird (or else we have to get into measure theory).
A specific example — periodic functions with period $2\pi$

To keep all those pesky $L$’s out of the way, let’s consider the space of (continuous, differentiable) functions that are periodic with period $2\pi$. As an exercise, show that all the functions in the set

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots\}$$

are perpendicular to one another. Also show that $|1| = \sqrt{2\pi}$, $|\sin nx| = \sqrt{\pi}$ and $|\cos nx| = \sqrt{\pi}$.

(This shows that this space of functions is infinite-dimensional.)

You can also show that for finite $N$ (and even for $N = \infty$ if the series converge), we have

$$\left| a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \right|^2 = 2\pi a_0 + \sum_{n=1}^{N} \pi(a_n^2 + b_n^2).$$
You probably remember the formula for the projection of the vector \( \mathbf{v} \) onto the vector \( \mathbf{w} \):

\[
\text{proj}_w(\mathbf{v}) = \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\langle \mathbf{w}, \mathbf{w} \rangle} \mathbf{w}.
\]

(You learned this in Math 114 – and it makes sense: it’s a vector of length \( |\mathbf{v}| \cos \theta \) in the direction of \( \mathbf{w} \)). You can generalize this as follows: If \( w_1, w_2, \ldots, w_N \) are orthogonal vectors (i.e., each is orthogonal to all of the others), then the projection of \( \mathbf{v} \) onto the subspace \( W \) spanned by the \( w_n \)'s is

\[
\text{proj}_W(\mathbf{v}) = \sum_{n=1}^{N} \frac{\langle \mathbf{v}, w_n \rangle}{\langle w_n, w_n \rangle} w_n.
\]
You should observe the correspondence between the projection formula
\[
\text{proj}_W(v) = \sum_{i=n}^{N} \frac{\langle v, w_n \rangle}{\langle w_n, w_n \rangle} w_n.
\]
and the Fourier series formula
\[
f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx + \sum_{n=1}^{N} \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \cos nx \\
+ \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \sin nx \\
= \frac{\langle f(x), 1 \rangle}{\langle 1, 1 \rangle} 1 + \sum_{n=1}^{N} \frac{\langle f(x), \cos nx \rangle}{\langle \cos nx, \cos nx \rangle} \cos nx + \frac{\langle f(x), \sin nx \rangle}{\langle \sin nx, \sin nx \rangle} \sin nx.
\]
The projection of $v$ onto the subspace $W$ is the closest vector in $W$ to $v$. You should check that the difference $v - \text{proj}_W(v)$ is perpendicular to $W$, and in particular it is perpendicular to $\text{proj}_W(v)$. Therefore the three vectors $\text{proj}_W(v)$, $v - \text{proj}_W(v)$ and $v$ are the sides of a right triangle, having $v$ as the hypotenuse. Thus,

$$|\text{proj}_W(v)|^2 + |v - \text{proj}_W(v)|^2 = |v|^2$$

by the Pythagorean theorem. Because squares are not negative, this results in Bessel’s inequality:

$$|\text{proj}_W(v)|^2 \leq |v|^2.$$
For Fourier series, we can rewrite Bessel’s inequality as

\[
2\pi a_0^2 + \sum_{n=1}^{N} \pi(a_n^2 + b_n^2) \leq |f|^2 = \int_{-\pi}^{\pi} (f(x))^2 \, dx
\]

Since this is true for all \(N\), we can let \(N \to \infty\) and obtain:

\[
2\pi a_0^2 + \sum_{n=1}^{\infty} \pi(a_n^2 + b_n^2) \leq |f|^2 = \int_{-\pi}^{\pi} (f(x))^2 \, dx
\]

Therefore, the series (of positive terms) on the left converges, and so \(a_n \to 0\) and \(b_n \to 0\) as \(n \to \infty\). This is the content of the Riemann-Lebesgue lemma:

\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = 0.
\]
A change in perspective

Now, let’s substitute the formula for the Fourier coefficients back into the $N$th partial sum $S_n(x)$ of the Fourier series:

$$S_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \, dy + \frac{1}{\pi} \sum_{n=1}^{N} \left( \int_{-\pi}^{\pi} f(y) \cos(ny) \, dy \right) \cos(nx)$$

$$+ \left( \int_{-\pi}^{\pi} f(y) \sin(ny) \, dy \right) \sin(nx)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \left( \frac{1}{2} + \sum_{n=1}^{N} \cos(ny) \cos(nx) + \sin(ny) \sin(nx) \, dy \right)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \left( \frac{1}{2} + \sum_{n=1}^{N} \cos(nx - ny) \right) \, dy$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) D_N(x - y) \, dy$$
The Dirichlet kernel

The function \( D_N(y) = \frac{1}{2} + \sum_{n=1}^{N} \cos(ny) \) from the last slide is called the \textit{Dirichlet kernel} of order \( N \).

Some properties of \( D_N(y) \) that are easy to prove are

- \( D_N \) is an even periodic function with period \( 2\pi \)
- \( \frac{1}{\pi} \int_{-\pi}^{\pi} D_N(y) \, dy = 1 \)

Using these and the equation \( S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y)D_N(x - y) \, dy \), we get (assuming \( f(x) \) has been extended as a periodic function of period \( 2\pi \)):

- \( S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x + y)D_N(y) \, dy \).
- \( S_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x - y)D_N(y) \, dy \).
- \( f(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)D_N(y) \, dy \).
A surprising sum

We can put the last three equations together to get a formula for how different the $N$th partial sum of the Fourier series is from $f(x)$:

$$S_N(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[ f(x + y) + f(x - y) - 2f(x) \right] D_N(y) \, dy.$$ 

To make use of this, we need the following surprising fact: We can evaluate the sum which defines $D_N(y)$!

$$D_N(y) = \frac{1}{2} + \sum_{n=1}^{N} \cos ny = \frac{\sin(N + \frac{1}{2})y}{2 \sin \frac{1}{2} y}.$$ 

(We’ll prove this in class using geometric sums and Euler’s formula — you’ll do another proof for homework that just uses trig identities)
Knowing the closed form of $D_N(y)$ allows us to write

\[
S_N(x) - f(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x + y) + f(x - y) - 2f(x)]D_N(y) \, dy
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x + y) + f(x - y) - 2f(x)] \frac{\sin(N + \frac{1}{2})y}{2 \sin \frac{1}{2} y} \, dy
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(x + y) + f(x - y) - 2f(x)}{2 \sin \frac{1}{2} y} \sin(N + \frac{1}{2})y \, dy
\]

So we’ll let

\[
Q(y) = \frac{f(x + y) + f(x - y) - 2f(x)}{2 \sin \frac{1}{2} y}
\]

and our job is to show that

\[
\lim_{N \to \infty} \int_{-\pi}^{\pi} Q(y) \sin(N + \frac{1}{2})y \, dy = 0.
\]
The necessary lemma

Lemma

If $Q(y)$ is a function such that $\int_{-\pi}^{\pi} (Q(y))^2 \, dy$ is finite, then

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} Q(y) \sin(N + \frac{1}{2})y \, dy = 0.$$

To prove this, note that

$$\sin((N + \frac{1}{2})y) = \sin Ny \cos \frac{1}{2}y + \cos Ny \sin \frac{1}{2}y.$$ So we can break the integral in the lemma into two:

$$\int_{-\pi}^{\pi} Q(y) \sin(N + \frac{1}{2})y \, dy = \int_{-\pi}^{\pi} (Q(y) \sin \frac{1}{2}y) \cos Ny \, dy + \int_{-\pi}^{\pi} (Q(y) \cos \frac{1}{2}y) \sin Ny \, dy$$

and use the Riemann-Lebesgue lemma on each.
To finish up, we need that

\[ Q(y) = \frac{f(x + y) + f(x - y) - 2f(x)}{2 \sin \frac{1}{2} y} \]

is square-integrable. This is no problem for reasonable \( f \) except where \( y = 0 \). But this is no problem where \( f \) is continuous and differentiable, nor is it a problem even at a point where \( f \) has a jump discontinuity, provided the derivatives from the left and right exist (such an \( f \) is called \textit{piecewise continuous}). At such points, the Fourier series will converge to the average of the left and right limits of \( f \).