Math 241: More heat equation/Laplace equation

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Another example

- Another heat equation problem:

\[ u_t = \frac{1}{2} u_{xx}, \quad u(0, t) = 0, \ u_x(L, t) = 0, \quad u(x, 0) = 2Lx - x^2 \]

for \( t > 0 \) and \( 0 \leq x \leq L \).
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for \( t > 0 \) and \( 0 \leq x \leq L \).

• To match the boundary conditions this time, we’ll assume that we can express \( u \) as

\[ u(x, t) = \sum_{n=0}^{\infty} b_n e^{-(n+\frac{1}{2})^2 \pi^2 t / 4 L^2} \sin \left( \frac{(n + \frac{1}{2})\pi x}{L} \right) \]

and see if we can figure out what the constants \( b_n \) should be — we know that the boundary conditions are automatically satisfied, and perhaps we can choose the \( b_n \)'s so that

\[ 2Lx - x^2 = \sum_{n=0}^{\infty} b_n \sin \left( \frac{(n + \frac{1}{2})\pi x}{L} \right). \]
Integral facts

- We’re trying to find $b_n$’s so that

$$2Lx - x^2 = \sum_{n=0}^{\infty} b_n \sin \left( \frac{n + \frac{1}{2}}{L} \pi x \right).$$
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\[
2Lx - x^2 = \sum_{n=0}^{\infty} b_n \sin \left( \frac{(n + \frac{1}{2})\pi x}{L} \right).
\]

• We’ll use two basic facts:

• If \( n \neq m \) then

\[
\int_0^L \sin \left( \frac{(n + \frac{1}{2})\pi x}{L} \right) \sin \left( \frac{(m + \frac{1}{2})\pi x}{L} \right) \, dx = 0.
\]

• If \( n = m \) then

\[
\int_0^L \sin \left( \frac{(n + \frac{1}{2})\pi x}{L} \right) \sin \left( \frac{(m + \frac{1}{2})\pi x}{L} \right) \, dx = \int_0^L \sin^2 \left( \frac{(n + \frac{1}{2})\pi x}{L} \right) \, dx = \frac{L}{2}.
\]
Finding the coefficients

• We’re still trying to find $b_n$’s so that

$$2Lx - x^2 = \sum_{n=0}^{\infty} b_n \sin \frac{(n + \frac{1}{2})\pi x}{L}.$$

• Motivated by the facts on the previous slide, we multiply both sides by $\sin \frac{(m_1)\pi x}{L}$ and integrate both sides from 0 to $L$:

$$\int_{0}^{L} (2Lx - x^2) \sin \frac{(m + \frac{1}{2})\pi x}{L} \, dx = \int_{0}^{L} \left( \sum_{n=0}^{\infty} b_n \sin \frac{(n + \frac{1}{2})\pi x}{L} \right) \sin \frac{(m + \frac{1}{2})\pi x}{L} \, dx$$

$$= \sum_{n=0}^{\infty} b_n \int_{0}^{L} \sin \frac{(n + \frac{1}{2})\pi x}{L} \sin \frac{(m + \frac{1}{2})\pi x}{K} \, dx$$

$$= \frac{Lb_m}{2}$$
Integration by parts

- It’s an exercise in integration by parts to show that

\[ \int_0^L (2Lx - x^2) \sin \left( \frac{m + \frac{1}{2}}{L} \pi x \right) dx = \frac{8L^3}{(m + \frac{1}{2})^3 \pi^3} \]
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- Therefore,

$$b_m = \frac{32L^2}{(2m + 1)^3\pi^3}$$
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• So we arrive at a candidate for the solution:

\[
u(x, t) = \sum_{n=0}^{\infty} \frac{32L^2}{(2n + 1)^3 \pi^3} e^{-\left(\frac{n + \frac{1}{2}}{2L^2}\right)^2 \pi^2 t} \sin \left( \frac{n + \frac{1}{2}}{L} \pi x \right)
\]
Validating the solution

- The series

\[ u(x, t) = \sum_{n=0}^{\infty} \frac{32L^2}{(2n + 1)^3 \pi^3} e^{-\left(\frac{n + \frac{1}{2}}{2L^2}\right)^2 \pi^2 t / 2L^2} \sin \left( \frac{n + \frac{1}{2}}{L} \pi x \right) \]

converges for \( t \geq 0 \), and certainly satisfies the boundary conditions. What about the initial condition \( u(x, 0) = 2Lx - x^2 \)?
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• Well,

\[ u(x, 0) = \sum_{n=0}^{\infty} \frac{32L^2}{(2n + 1)^3 \pi^3} \sin \left( \frac{(n + \frac{1}{2})\pi x}{L} \right) \]
Graphical evidence

Use $L = 5$. Red graph: $10x - x^2$, Blue graph: sum

One term:

Three terms:
Plotting the solution

Here is a plot of the sum of the first three terms of the solution:
Laplace equation on a rectangle

• The two-dimensional Laplace equation is

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• We’ll solve the equation on a bounded region (at least at first), and it’s appropriate to specify the values of \( u \) on the boundary (Dirichlet boundary conditions), or the values of the normal derivative of \( u \) at the boundary (Neumann conditions), or some mixture of the two.
One uniqueness proof: suppose $u = 0$ on the boundary of the region (so $u$ could arise as the difference between two solutions of the Dirichlet problem).
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Since \( \nabla \cdot (u \nabla u) = u \nabla \cdot \nabla u + (\nabla u) \cdot (\nabla u) \), the divergence theorem tells us:

\[
\int_{\partial R} u \nabla u \cdot n \, ds - \int_R \nabla^2 u \, dA.
\]

But the right side is zero because \( u = 0 \) on \( \partial R \) (the boundary of \( R \)) and because \( \nabla^2 u = 0 \) throughout \( R \).

So we conclude \( u \) is constant, and thus zero since \( u = 0 \) on the boundary.
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Case 1: \( R \) is a rectangle

- We’ll first solve the Dirichlet problem on a rectangle, say \( 0 \leq x \leq L \) and \( 0 \leq y \leq M \).
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- Let’s see what we get by separation of variables: Let \( u(x, y) = X(x)Y(y) \), then \( X''Y + XY'' = 0 \), or

\[
\frac{X''}{X} = -\frac{Y''}{Y} = \lambda
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a constant because \( X'''/X \) is a function of \( x \) alone and \( Y''/Y \) is a function of \( y \) alone.
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$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

a constant because $X''''/X$ is a function of $x$ alone and $Y''/Y$ is a function of $y$ alone.
- The boundary data might look like:

$$u(x, 0) = f(x), \quad u(L, y) = g(y)$$

$$u(x, M) = h(x), \quad u(0, y) = k(y)$$

but that’s too much to deal with all at once.
• You can add together four solutions to problems with boundary data like:

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Since \( u \) is zero for two values of \( y \), we’ll use \( Y'' + \alpha^2 Y = 0 \), so \( X'' - \alpha^2 X = 0 \).

This gives

\[ Y = \sin \left( \frac{n\pi y}{M} \right) \quad n = 1, 2, 3 \ldots \]

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A useful way to write the corresponding \( X \) solutions is

\[ X = \sinh \left( \frac{n\pi}{M} (L - x) \right) \]

so that \( X(L) = 0 \).
So we now have

\[ u(x, y) = \sum_{n=1}^{\infty} b_n \sinh \left( \frac{n\pi}{M} (L - x) \right) \sin \left( \frac{n\pi y}{M} \right). \]

and we have to match:

\[ u(0, y) = k(y) = \sum_{n=1}^{\infty} b_n \sinh \left( \frac{n\pi L}{M} \right) \sin \left( \frac{n\pi y}{M} \right). \]
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Thus

$$b_n = \frac{2}{M \sinh \left( \frac{n\pi L}{M} \right)} \int_{0}^{M} k(y) \sin \left( \frac{n\pi y}{M} \right) \, dy.$$
Laplace on a disk

• Next up is to solve the Laplace equation on a disk with boundary values prescribed on the circle that bounds the disk.

• We’ll use polar coordinates for this, so a typical problem might be:

\[ \nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = 0 \]

on the disk of radius \( R = 3 \) centered at the origin, with boundary condition

\[ u(3, \theta) = \begin{cases} 
1 & 0 \leq \theta \leq \pi \\
\sin^2 \theta & \pi < \theta < 2\pi 
\end{cases} \]