• View Locust Walk between 36th and 38th Streets as one-dimensional, about 200m long.
• Take $x = 0$ at 36th Street and $x = 200$ at base of bridge at 38th Street.
• Let $u(x, t)$ be the density of squirrels (in squirrels per meter) at time $t$ and at position $x$. 
• Assume squirrels are neither created nor destroyed, and that they don’t leave the Walk going north or south.
Conservation of squirrels

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• Let $\varphi(x, t)$ be the flux of squirrels past $x$ at time $t$. This is the net rate at which squirrels are moving toward larger values of $x$ (i.e., west).
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- Let $\varphi(x, t)$ be the flux of squirrels past $x$ at time $t$. This is the net rate at which squirrels are moving toward larger values of $x$ (i.e., west).
- If $N_{a,b}(t)$ is the number of squirrels between $x = a$ and $x = b$ at time $t$ (for $0 \leq a < b \leq 200$), then we have

$$N_{a,b}(t) = \int_a^b u(x, t) \, dx.$$
• Under these assumptions, $N_{a,b}$ can only change because squirrels move past $x = a$ or past $x = b$ — so
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• Of course, we also have

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Use the fundamental theorem of calculus to conclude that

$$\frac{\partial u}{\partial t} = -\frac{\partial \varphi}{\partial x}$$

for all $x$ and $t$. 
Sciurine constitution

- Whereas the conservation law is purely mathematical, we’ll need some assumption about how the squirrels move to get an equation for $u(x, t)$.

- Empirically, we observe that when squirrels encounter each other, usually one of them chases the other away — so their general tendency is to spread out, and move from a region of high squirrel density to one of lower density.

- So the flux $\phi(x, t)$ will be positive if the density is higher for $x < a$ than for $x > a$, i.e., if $\frac{\partial u}{\partial x} < 0$, and vice versa.

- The simplest “constitutive law” that expresses this is for $\phi(x, t)$ to be proportional to $\frac{\partial u}{\partial x}$ with a negative proportionality constant, so we’ll write:

$$\phi(x, t) = -k \frac{\partial u}{\partial x}.$$
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Now, we’ll put our conservation equation

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together with our constitutive equation

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and eliminate $\varphi$ to get

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}.$$
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This is the *diffusion* (heat) equation for squirrels.

Replace squirrel density with density of a contaminant or solute or other population to get diffusion in other circumstances.
Initial and boundary conditions

- In order to have a well-determined solution, we’ll need to specify two kinds of conditions:

  - Since the equation is first-order in $t$, we expect to specify the initial state, i.e., a condition such as $u(x, t) = f(x)$ for a given function $f(x)$.

  - We also have to know what goes on at the ends of the walk, i.e., no flux at 38th street (which would be $u_x(200, t) = 0$). Also, suppose we know that the density of squirrels at 36th street is 0.5 squirrels per meter so $u(0, t) = 0.5$.

- We’ll prove uniqueness, and worry about existence later.
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• We’ll prove uniqueness, and worry about existence later.
• We’ll prove that there is only one solution to the initial/boundary value problem for the heat equation:
  • \( u_t = ku_{xx} \) for \( t > 0 \) and \( x \in [a, b] \)
  • \( u(a, t) = f(t) \) (or \( u_x(a, t) = f(t) \))
    and \( u(b, t) = g(t) \) (or \( u_x(b, t) = g(t) \)) for \( t > 0 \)
  • \( u(x, 0) = h(x) \) for \( x \in [a, b] \)

• To do this, assume there are two solutions, \( u_1 \) and \( u_2 \), and let \( v = u_1 - u_2 \) (as usual). Then \( v \) satisfies the problem:
  • \( v_t = kv_{xx} \) for \( t > 0 \) and \( x \in [a, b] \)
  • \( v(a, t) = 0 \) (or \( v_x(a, t) = 0 \))
    and \( v(b, t) = 0 \) (or \( v_x(b, t) = 0 \)) for \( t > 0 \)
  • \( v(x, 0) = 0 \) for \( x \in [a, b] \)
Uniqueness, continued

- Let $E(t) = \int_a^b (v(x, t))^2 \, dx$. Clearly $E(t) \geq 0$ for all $t$ and $E(0) = 0$.
- Calculate:
  \[
  \frac{dE}{dt} = \int_a^b 2vv_t \, dx = \int_a^b 2kvv_x \, dx.
  \]
- Integrate by parts:
  \[
  \frac{dE}{dt} = 2kvv_x \bigg|_{x=a}^{x=b} - \int_a^b (v_x)^2 \, dx.
  \]
- But the first term is zero by the boundary conditions, and the second one is not positive.
- So we must have $E(t) = 0$ for all $t$, which implies $v(x, t) = 0$ for all $x$ and $t$, and we’re done.
We can think of the College green as roughly a square, 50 meters on a side, running from 34th to 36th Streets and from Walnut to Perelman Quad.

The new element in two dimensions is that flux is a little trickier to define.

Flux is a vector, and now its units are squirrels per unit length per unit time.
Flux in two dimensions

- Let $R$ be a region in the plane bounded by the curve $C$, with outward pointing normal vector $n$. 
Flux in two dimensions

- Let $R$ be a region in the plane bounded by the curve $C$, with outward pointing normal vector $n$.

- At a point of the boundary, the local flux across it is $\varphi \cdot n \, ds$ (in squirrels per second).
Continuity (conservation) equation

• Adding all this together gives the total (outward) flux as

\[ \int_C \varphi \cdot \mathbf{n} \, ds. \]
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- So the rate of change of squirrels in \( R \) is
  \[
  \frac{\partial}{\partial t} \int \int_R u \, dA = \int \int_R \frac{\partial u}{\partial t} \, dA = - \int C \varphi \cdot \mathbf{n} \, ds = - \int \int_R \nabla \cdot \varphi \, dA
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  by the divergence theorem.
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by the divergence theorem.

- Since this holds for all regions \( R \) we have at each point:

\[ \frac{\partial u}{\partial t} = -\nabla \cdot \varphi. \]

This is usually called the continuity equation.
Constitutive equation

- As before, we’ll assume the squirrels generally want to get away from each other.
- This means they want to go from higher-density areas to lower-density ones.

\[ \phi = -k \nabla u \]

Put this together with the continuity equation

\[ \frac{\partial u}{\partial t} = -\nabla \cdot \phi \]

and get the two-dimensional diffusion equation

\[ \frac{\partial u}{\partial t} = k \nabla^2 u \]

The quantity on the right is usually written \( \nabla^2 u \) (by physicists and engineers) or \( \Delta u \) (by mathematicians) and is called the Laplacian of \( u \).
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In other words, the flux of squirrels will go against the gradient of the squirrel density, i.e.,

$$\varphi = -k \nabla u.$$ 

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Squirrels in space! The 3D heat equation

- The 3D diffusion equation works just like the 2D one, except you have a solid region in 3D-space, the flux vector is a 3D-vector, and flux has units of squirrels per unit area per unit time.

- The conservation and constitutive equations still look the same, and the diffusion equation is still

\[
\frac{\partial u}{\partial t} = k \nabla^2 u.
\]