Hints to homework 8

10.4.3. The problem is
\[ u_t = ku_{xx} + cu_x, \quad u(x, 0) = f(x). \]

(a) We take the Fourier transform of both sides with respect to \(x\) and get
\[ \hat{u}_t = -k\omega^2 \hat{u} - ic\omega \hat{u}, \quad \hat{u}(\omega, 0) = \hat{f}(\omega). \]
The differential equation has become an ordinary differential equation with independent variable \(t\):
\[ \hat{u}_t + (k\omega^2 + ic\omega)\hat{u} = 0, \]
which has general solution
\[ \hat{u}(\omega, t) = a(\omega)e^{-(k\omega^2 + ic\omega)t} \]
and since
\[ \hat{u}(\omega, 0) = a(\omega) = \hat{f}(\omega), \]
we have
\[ \hat{u}(\omega, t) = \hat{f}(\omega)e^{-(k\omega^2 + ic\omega)t}. \]
Therefore,
\[ u(x, t) = \mathcal{F}^{-1} \left[ \hat{f}(\omega)e^{-(k\omega^2 + ic\omega)t} \right] (x, t). \]
\[ = \mathcal{F}^{-1} \left[ e^{ict\omega} \left( \hat{f}(\omega)e^{-kt\omega^2} \right) \right] (x, t) \]
\[ = \mathcal{F}^{-1} \left[ \hat{f}(\omega)e^{-kt\omega^2} \right] (x - ct, t) \]
\[ = \frac{1}{2\pi} \left( f(x) * \sqrt{\frac{\pi}{kt}}e^{-x^2/(4kt)} \right) (x - ct, t) \]
\[ = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y)e^{-(x-ct-y)^2/(4kt)} dy \]

(b) The solution is the same as the solution of the heat equation except translated to the right by \(ct\) because of the convection term.

10.4.4. The problem is
\[ u_t = ku_{xx} - \gamma u, \quad u(x, 0) = f(x). \]

(a) The usual drill: take the Fourier transform with respect to \(x\):
\[ \hat{u}_t = -k\omega^2 \hat{u} - \gamma \hat{u}, \quad \hat{u}(\omega, 0) = \hat{f}(\omega). \]
Solve the ODE (in the variable \(t\), treating \(\omega\) as a parameter)
\[ \hat{u}_t + (k\omega^2 + \gamma)\hat{u} = 0, \quad \hat{u}(\omega, 0) = \hat{f}(\omega) \]
and get
\[ \hat{u}(\omega, t) = \hat{f}(\omega)e^{-(k\omega^2 + \gamma)t}. \]
Therefore
\[ u(x, t) = \mathcal{F}^{-1} \left[ \hat{f}(\omega)e^{-(k\omega^2+\gamma)t} \right](x, t) \]
\[ = \mathcal{F}^{-1} \left[ (e^{-\gamma t} \hat{f}(\omega)) * e^{-kt\omega^2} \right] (x, t) \]
\[ = \frac{1}{2\pi} \left( e^{-\gamma t} f(x) * \sqrt{\frac{\pi}{kt}} e^{-x^2/(4kt)} \right) \]
\[ = \frac{e^{-\gamma t}}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} f(y)e^{-(x-y)^2/(4kt)} dy. \]

(b) The solution is just \( e^{-\gamma t} \) times the solution of the ordinary heat equation. If we had set \( v(x, t) = e^{\gamma t} u(x, t) \) at the beginning, this would have transformed the problem into
\[ v_t = kv_{xx}, \quad v(x, 0) = f(x), \]
which is a problem we already knew the solution of.

**10.4.7.** This time the problem is
\[ \hat{u} = ku_{xxx}, \quad u(x, 0) = f(x). \]

(a) Take the Fourier transform with respect to \( x \) (noting that \( i^3 = -i \)):
\[ \hat{u}_t = ik\omega^3 \hat{u}, \quad \hat{u}(\omega, 0) = \hat{f}(\omega). \]
The solution of the ODE
\[ \hat{u}_t - ik\omega^3 \hat{u} = 0, \quad \hat{u}(\omega, 0) = \hat{f}(\omega) \]
is
\[ \hat{u}(\omega, t) = \hat{f}(\omega)e^{ik\omega^3 t}. \]
Therefore
\[ u(x, t) = \mathcal{F}^{-1} \left[ \hat{f}(\omega)e^{ik\omega^3} \right](x, t) \]
\[ = \frac{1}{2\pi} \left( f(x) * \mathcal{F}^{-1} \left[ e^{ik\omega^3} \right] (x, t) \right). \]
So now we have to think about the inverse Fourier transform of \( e^{ik\omega^3} \).

The problem mentions looking at the preceding problem (10.4.6), which concerns the solution (known as the *Airy function* \( \text{Ai}(x) \)) of the ordinary differential equation
\[ \frac{d^2y}{dx^2} - xy = 0, \quad y(0) = \frac{1}{\pi} \int_0^\infty \cos \left( \frac{s^3}{3} \right) ds. \]
We solve this ODE by taking Fourier transforms and get
\[ -\omega^2 \hat{y} + \frac{d\hat{y}}{d\omega} = 0. \]
or
\[
\frac{d\hat{y}}{d\omega} = -i\omega^2 \hat{y}.
\]
This is a separable differential equation (note, the independent variable is \(\omega\)!), and so
\[
\frac{d\hat{y}}{\hat{y}} = -i\omega^2 d\omega
\]
or
\[
\ln(\hat{y}(\omega)) = \ln(C) - \frac{i}{3}\omega^3.
\]
which is the same as
\[
\hat{y}(\omega) = Ce^{-i\omega^3/3}.
\]
So we can take the inverse transform of both sides and get
\[
y(x) = C \int_{-\infty}^{\infty} e^{-i\omega^3/3} \cos\left(\frac{\omega^3}{3} + x\omega\right) d\omega.
\]
We divide the interval of integration in half and write
\[
y(x) = C \left( \int_{-\infty}^{0} e^{-i\omega^3/3} \cos\left(\frac{\omega^3}{3} + x\omega\right) d\omega + \int_{0}^{\infty} e^{-i\omega^3/3} \cos\left(\frac{\omega^3}{3} + x\omega\right) d\omega \right)
\]
and then make the substitution \(\eta = -\omega\) on the \(-\infty\) to 0 half:
\[
y(x) = C \left( \int_{0}^{\infty} e^{i\eta^3/3} \cos\left(\frac{\eta^3}{3} + x\eta\right) d\eta + \int_{0}^{\infty} e^{-i\omega^3/3} \cos\left(\frac{\omega^3}{3} + x\omega\right) d\omega \right).
\]
We can now replace \(\eta\) with \(\omega\) in the first integral and combine the two to get
\[
y(x) = 2C \int_{0}^{\infty} \cos\left(\frac{\omega^3}{3} + x\omega\right) d\omega.
\]
Note that
\[
y(0) = 2C \int_{0}^{\infty} \cos\left(\frac{\omega^3}{3}\right) d\omega
\]
so according to problem 10.4.6 in the text we should set \(2C = 1/\pi\) to get the Airy function
\[
\text{Ai}(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos\left(\frac{\omega^3}{3} + x\omega\right) d\omega
\]
and we will have
\[
\widehat{\text{Ai}}(\omega) = \frac{1}{2\pi} e^{-i\omega^3/3}
\]
(note that if you look this up on Wikipedia, the Fourier transform of \(\text{Ai}(x)\) is a little different because they use a different definition of the Fourier transform).

(b)(c) Now we can return to the problem for the KdV equation: we needed the inverse Fourier transform of \(e^{ikt\omega^3}\). We can use our formula for the Fourier transform of the Airy function together with the scaling rule for the Fourier transform:
\[
\mathcal{F}\left[\frac{1}{a} f\left(\frac{x}{a}\right)\right] = \hat{f}(a\omega)
\]
with \( a = -(3kt)^{1/3} \) to get

\[
\mathcal{F} \left[ 2\pi \frac{-1}{(3kt)^{1/3}} \text{Ai} \left( -\frac{x}{(3kt)^{1/3}} \right) \right] (\omega) = 2\pi \text{Ai} \left( -(3kt)^{1/3} \omega \right) = e^{-i[-(3kt)^{1/3} \omega]^{3/3}} = e^{ikt\omega^3}
\]

or

\[
\mathcal{F}^{-1} \left[ e^{ikt\omega^3} \right] = 2\pi \frac{-1}{(3kt)^{1/3}} \text{Ai} \left( \frac{-x}{(3kt)^{1/3}} \right).
\]

So we have

\[
F(x, t) = \frac{1}{2\pi} \left( f(x) * \mathcal{F}^{-1} \left[ e^{ikt\omega^3} \right] (x, t) \right)
= f(x) * \frac{-1}{(3kt)^{1/3}} \text{Ai} \left( \frac{-x}{(3kt)^{1/3}} \right)
= \frac{-1}{(3kt)^{1/3}} \int_{-\infty}^{\infty} f(y) \text{Ai} \left( \frac{-x-y}{(3kt)^{1/3}} \right) dy.
\]

**10.6.3.** The problem is to solve the Laplace equation

\[
u_{xx} + u_{yy} = 0 \quad \text{for} \quad x < 0, \quad -\infty < y < \infty,
\]

i.e., in the left half-plane, subject to the boundary condition

\[u(0, y) = g(y).
\]

(a) We can take the Fourier transform with respect to \( y \) (treat \( x \) as a parameter) and find that

\[
\hat{u}_{xx} - \omega^2 \hat{u} = 0, \quad \hat{u}(0, \omega) = \hat{g}(\omega).
\]

Treat this as an ordinary differential equation with \( x \) as the independent variable and get

\[
\hat{u}(x, \omega) = c_1(\omega)e^{\omega x} + c_2(\omega)e^{-\omega x} \quad \text{if} \quad \omega \neq 0
\]

and

\[
\hat{u}(x, 0) = c_1(0) + c_2(0) \omega.
\]

This gets a little bit tricky because \( x \) is negative — Implicit in the problem is the condition that \( u \) should remain bounded (certainly \( \hat{u} \) needs to remain bounded if we’re going to take its inverse Fourier transform), so (keeping in mind that \( x \) is negative) we can declare that \( c_2(\omega) = 0 \) if \( \omega > 0 \) and \( c_1(\omega) = 0 \) if \( \omega \leq 0 \). So we have

\[
\hat{u}(x, \omega) = \begin{cases} 
\hat{g}(\omega)e^{\omega x} & \text{if} \quad \omega > 0 \\
\hat{g}(0) & \text{if} \quad \omega = 0 \\
\hat{g}(\omega)e^{-\omega x} & \text{if} \quad \omega < 0
\end{cases} = \hat{g}(\omega)e^{x|\omega|}.
\]

We conclude that (also remember that the variable we Fourier transformed was \( y \) and \( x \) is a parameter here)

\[
u(x, y) = \frac{1}{2\pi} \left( g(y) * \mathcal{F}^{-1} \left[ e^{x|\omega|} \right] \right) = \frac{1}{2\pi} \left( g(y) * \frac{-2x}{x^2 + y^2} \right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s) \frac{-2x}{x^2 + (y - s)^2} ds
\]
(b) If \( g(y) = S_1(y) \) then the solution becomes
\[
  u(x, y) = \frac{1}{2\pi} \int_{-1}^{1} \frac{-2x}{x^2 + (y - s)^2} \, ds.
\]
If we let \( u = (s - y)/x \) (or \( s = xu + y \), so \( ds = x \, du \)) then we have
\[
  u(x, y) = -\frac{1}{\pi} \int_{(-1-y)/x}^{(1-y)/x} \frac{du}{1 + u^2} = -\frac{1}{\pi} \arctan \left( \frac{(1-y)/x}{(-1-y)/x} \right) = \frac{1}{\pi} \left( \arctan \left( \frac{y-1}{x} \right) - \arctan \left( \frac{y+1}{x} \right) \right)
\]
since arctangent is an odd function.

10.6.9. The problem is
\[
  u_t = k \nabla^2 u - v_0 \cdot \nabla u, \quad u(x, y, 0) = f(x, y).
\]
If we write \( v = [a, b] \) then the equation becomes
\[
  u_t = k (u_{xx} + u_{yy}) - au_x - bu_y.
\]
Taking our cue from problem 10.4.3, we set
\[
  u(x, y, t) = v(x - at, y - bt, t) \quad \text{or} \quad v(x, y, t) = u(x + at, y + bt, t)
\]
Then we will have
\[
  v_t(x, y, t) = u_t(x + at, y + bt, t) + au_x(x + at, y + bt, t) - bu_y(x + at, y + bt, t)
\]
\[
  v_x(x, y, t) = u_x(x + at, y + bt, t)
\]
\[
  v_{xx}(x, y, t) = u_{xx}(x + at, y + bt, t)
\]
\[
  v_y(x, y, t) = u_y(x + at, y + bt, t)
\]
\[
  v_{yy}(x, y, t) = u_{yy}(x + at, y + bt, t)
\]
From these equations together with the differential equation \( u_t = k (u_{xx} + u_{yy}) - au_x - bu_y \) we conclude
\[
  v_t = k (v_{xx} + v_{yy}), \quad \text{and} \quad v(x, y, 0) = f(x, y)
\]
so \( v \) satisfies the two-dimensional heat equation.

To solve this, we can either take Fourier transforms one space variable at a time (i.e., first in \( x \) and then in \( y \)), or use a two-dimensional Fourier transform:
\[
  \mathcal{F} [f(x, y)] (\omega_1, \omega_2) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{i(\omega_1 x + \omega_2 y)} \, dx \, dy.
\]
If we do this to both sides of the equation we obtain
\[
  \hat{v}_t(\omega_1, \omega_2, t) = -k(\omega_1^2 + \omega_2^2) \hat{v}(\omega_1, \omega_2, t), \quad \hat{v}(\omega_1, \omega_2, 0) = \hat{f}(\omega_1, \omega_2).
\]
This is an ordinary differential equation in \( v \) with solution
\[
  \hat{v}(\omega_1, \omega_2, t) = \hat{f}(\omega_1, \omega_2) e^{-k(\omega_1^2 + \omega_2^2)t}.
\]
The inverse Fourier transform of the right side is a “two-dimensional convolution” (and we’ll derive the Fourier transform of the Gaussian at the end so as not to interrupt the flow of the story):

\[ v(x, y, t) = \frac{1}{4\pi^2} f(x, y) * \mathcal{F}^{-1}\left[e^{-k(\omega_1^2 + \omega_2^2)t}\right] \]

\[ = \frac{1}{4\pi^2} f(x, y) * \frac{\pi}{kt} e^{-(x^2 + y^2)/(4kt)} \]

\[ = \frac{1}{4\pi kt} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) e^{-[(x-r)^2 + (y-s)^2]/(4kt)} \, dr \, ds \]

Finally, we have

\[ u(x, y, t) = v(x - at, y - bt, t) = \frac{1}{4\pi kt} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(r, s) e^{-[(x-at-r)^2 + (y-bt-s)^2]/(4kt)} \, dr \, ds. \]

We finish up by deriving the Fourier transform of the two-variable Gaussian. Recall that we know that for the single variable Gaussian

\[ \mathcal{F}\left[e^{-a x^2/2}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a x^2/2} e^{ix\omega} \, dx = \frac{1}{\sqrt{2\pi a}} e^{-\omega^2/(2a)}. \]

So we can do the two-dimensional version one dimension at a time:

\[ \mathcal{F}\left[e^{-a(x^2+y^2)/2}\right] = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \mathcal{F}\left[e^{-a x^2/2}\right] (\omega_1) \right) e^{-a y^2/2} e^{-iy\omega_2} \, dy \]

\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a x^2/2} e^{ix\omega_1} \, dx \right) e^{-a y^2/2} e^{-iy\omega_2} \, dy \]

\[ = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a x^2/2} e^{ix\omega_1} \, dx \right) \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-a y^2/2} e^{iy\omega_2} \, dy \right) \]

\[ = \left( \frac{1}{\sqrt{2\pi a}} e^{-\omega_1^2/(2a)} \right) \left( \frac{1}{\sqrt{2\pi a}} e^{-\omega_2^2/(2a)} \right) \]

\[ = \frac{1}{2\pi a} e^{-(\omega_1^2 + \omega_2^2)/(2a)}. \]

So in particular, to get \( e^{-k(\omega_1^2 + \omega_2^2)t} \) as the Fourier transform, we must have \( 1/(2a) = kt \), or \( a = 1/(2kt) \). Therefore

\[ \mathcal{F}^{-1}\left[e^{-k(\omega_1^2 + \omega_2^2)t}\right] = \frac{2\pi}{2kt} e^{-(x^2+y^2)/(4kt)}. \]

10.6.18. The problem is

\[ u_{tt} = c^2 u_{xx} \quad u(x, 0) = 0 \quad u_t(x, 0) = g(x). \]

We take the Fourier transform of both sides (in \( x \)) and get

\[ \hat{u}_{tt} = -c^2 \omega^2 \hat{u}, \quad \hat{u}(\omega, 0) = 0 \quad \hat{u}_t(\omega, 0) = \hat{g}(t). \]

This ordinary differential equation has solution

\[ \hat{u} = \frac{\hat{g}(\omega)}{c\omega} \sin(c\omega t) = \frac{\sin(c\omega t)}{c\omega}. \]
Recalling that $\hat{S}_a(\omega) = \sin a\omega/(\pi \omega)$ and that the inverse Fourier transform of a product of two functions is their convolution (divided by $2\pi$), we have

$$u(x,t) = \mathcal{F}^{-1} \left[ \hat{g}(\omega) \frac{\sin \omega ct}{\omega} \right]$$

$$= \frac{1}{2\pi} g(x) * \left( \frac{\pi}{c} S_{ct}(x) \right)$$

$$= \frac{1}{2c} \int_{-\infty}^{\infty} g(x - y) S_{ct}(y) \, dy$$

$$= \frac{1}{2c} \int_{-ct}^{ct} g(x - y) \, dy$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} g(u) \, du$$

where in the last step we made the substitution $u = x - y$ (so $y = x - u$ and $dy = -du$).

**Part II – problems from the homework sheet.**

(a) From the formula for the Fourier transform of the Gaussian, we already know that

$$\hat{h}_0(\omega) = \mathcal{F} \left[ e^{-x^2/2} \right] (\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/2} = \frac{i^0}{\sqrt{2\pi}} h_0(\omega).$$

Now, suppose we already know for some value of $n$ that

$$\hat{h}_n(\omega) = \frac{i^n}{\sqrt{2\pi}} h_n(\omega).$$

Then

$$\hat{h}_{n+1}(\omega) = \mathcal{F} \left[ x h_n(x) - h'_n(x) \right] (\omega)$$

$$= -i\omega \hat{h}_n(\omega) + i\omega \hat{h}_n(\omega)$$

$$= i \left( \omega \hat{h}_n(\omega) - \frac{d}{d\omega} \hat{h}_n(\omega) \right)$$

$$= i \left( \frac{i^n}{\sqrt{2\pi}} \omega h_n(\omega) - \frac{i^n}{\sqrt{2\pi}} h'_n(\omega) \right)$$

$$= \frac{i^{n+1}}{\sqrt{2\pi}} (\omega h_n(\omega) - h'_n(\omega))$$

$$= \frac{i^{n+1}}{\sqrt{2\pi}} h_{n+1}(\omega).$$

We have shown that if the statement of the problem is true for $n$ then it is true for $n + 1$, and we know it’s true for $n = 0$. Therefore the statement is true for all $n \geq 0$ by mathematical induction.

(b) We can also prove inductively that $h_n(x)$ is a polynomial $H_n(x)$ of degree $n$ times $e^{-x^2/2}$. Since

$$h_0(x) = e^{-x^2/2} = 1 \cdot e^{-x^2/2},$$

we have $H_0(x) = 1$ so the statement is true for $n = 0$. 

Now assume the statement is true for some value of \( n \), so
\[
h_n(x) = H_n(x)e^{-x^2/2}
\]
with \( H_n(x) \) a polynomial of degree \( n \). Then
\[
h_{n+1}(x) = xh_n(x) - H'_n(x)
\]
\[
= xH_n(x)e^{-x^2/2} - (H_n(x)e^{-x^2/2})'
\]
\[
= xH_n(x)e^{-x^2/2} - (H'_n e^{-x^2/2} - xH_ne^{-x^2/2})
\]
\[
= (2xH_n(x) - H'_n(x))e^{-x^2/2}
\]
\[
= H_{n+1}(x)e^{-x^2/2},
\]
where \( H_{n+1}(x) = 2xH_n(x) - H'_n(x) \). And since \( H_n(x) \) is a polynomial of degree \( n \), we have \( 2xH_n(x) \) is a polynomial of degree \( n + 1 \) and \( H'_n(x) \) is a polynomial of degree \( n - 1 \), so their difference is a polynomial of degree \( n + 1 \). (We can also see that the leading coefficient, i.e., the coefficient of \( x^n \), in \( H_n(x) \) is \( 2^n \).)

(c) Since \( H_{n+1}(x) = 2xH_n - H'_n(x) \), we have
\[
H_0(x) = 1
\]
\[
H_1(x) = 2xH_0(x) - H'_0(x) = 2x(1) - 0 = 2x
\]
\[
H_2(x) = 2xH_1(x) - H'_1(x) = 2x(2x) - 2 = 4x^2 - 2
\]
\[
H_3(x) = 2xH_2(x) - H'_2(x) = 2x(4x^2 - 2) - (8x) = 8x^3 - 12x
\]
\[
H_4(x) = 2xH_3(x) - H'_3(x) = 2x(8x^3 - 12x) - (24x^2 - 12) = 16x^4 - 48x^2 + 12
\]
\[
H_5(x) = 2xH_4(x) - H'_4(x) = 2x(16x^4 - 48x^2 + 12) - (64x^3 - 96x) = 32x^5 - 160x^3 + 120x
\]
\[
H_6(x) = 2xH_5(x) - H'_5(x) = 2x(32x^5 - 160x^3 + 120x) - (160x^4 - 480x^2 + 120)
\]
\[
= 64x^6 - 480x^4 + 720x^2 - 120
\]

(d) We know that \( xh_n(x) - H'_n(x) = h_{n+1}(x) \) and that \( h_0(x) = e^{-x^2/2} \). We’ll begin by proving a “closed form” formula for \( h_n \) as follows. If we look at the equation
\[
H'_n(x) - xh_n(x) = -h_{n+1}(x)
\]
as though it were a first-order linear differential equation for \( h_n \), then the integrating factor would be \( e^{-x^2/2} \), so we multiply through by \( e^{-x^2/2} \) and get
\[
\frac{d}{dx} \left( e^{-x^2/2}h_n(x) \right) = -e^{-x^2/2}h_{n+1}(x),
\]
or
\[
h_{n+1}(x) = -e^{x^2/2} \frac{d}{dx} \left( e^{-x^2/2}h_n(x) \right).
\]
If we write out the first few \( h_n \)'s this way, we see that
\[
h_0(x) = e^{-x^2/2} = e^{x^2/2}e^{-x^2}
\]
\[
h_1(x) = -e^{x^2/2} \frac{d}{dx} \left( e^{-x^2/2}h_0(x) \right) = -e^{x^2/2} \frac{d}{dx} \left( e^{-x^2} \right)
\]
\[
h_2(x) = -e^{x^2/2} \frac{d}{dx} \left( e^{-x^2/2}h_1(x) \right) = e^{x^2/2} \frac{d^2}{dx^2} \left( e^{-x^2} \right)
\]
This leads us to conjecture that
\[ h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2} \right). \]

We’ve already seen that this is true for \( n = 0, 1, 2 \) and we now suppose it is true for a certain value of \( n \) and attempt to prove that it’s true for \( n + 1 \). So suppose that
\[ h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2} \right). \]

We know that
\[ h_{n+1}(x) = x h_n(x) - h'_n(x) \]
\[ = - (h'_n(x) - x h_n(x)) \]
\[ = - e^{x^2/2} \left( e^{-x^2/2} h'_n(x) - x e^{-x^2/2} h_n(x) \right) \]
\[ = - e^{x^2/2} \frac{d}{dx} \left( e^{-x^2/2} h_n(x) \right) \]
\[ = - e^{x^2/2} \frac{d}{dx} \left( e^{-x^2/2} \left( (-1)^n e^{-x^2/2} \frac{d^n}{dx^n} e^{-x^2} \right) \right) \]
\[ = (-1)^{n+1} e^{x^2/2} \frac{d^{n+1}}{dx^{n+1}} \left( e^{-x^2} \right), \]
which is what we were trying to show. The formula
\[ h_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} \left( e^{-x^2} \right) \]
is called Rodrigues’s formula for the Hermite functions.

Now that we have a fairly explicit formula for \( h_n \) we should be able to prove the new identity, but we’ll need one more little factoid: For any function \( f(x) \),
\[ \frac{d^n}{dx^n} (xf(x)) = n \frac{d^{n-1}}{dx^{n-1}} (f(x)) + x \frac{d^n}{dx^n} (f(x)). \]
This is certainly true if \( n = 0 \), and if it’s true for \( n \) then
\[ \frac{d^{n+1}}{dx^{n+1}} (xf(x)) = \frac{d}{dx} \left( \frac{d^n}{dx^n} (xf(x)) \right) \]
\[ = \frac{d}{dx} \left( n \frac{d^{n-1}}{dx^{n-1}} (f(x)) + x \frac{d^n}{dx^n} (f(x)) \right) \]
\[ = n \frac{d^n}{dx^n} (f(x)) + \frac{d^n}{dx^n} (f(x)) + x \frac{d^{n+1}}{dx^{n+1}} (f(x)) \]
\[ = (n + 1) \frac{d^n}{dx^n} + x \frac{d^{n+1}}{dx^{n+1}} (f(x)), \]
so it’s true for \( n + 1 \) and we’ve proved the factoid by induction.
Next, use the factoid to note that
\[
\frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = \frac{d^n}{dx^n} \left( \frac{d}{dx} (e^{-x^2}) \right)
\]
\[
= \frac{d^n}{dx^n} (-2xe^{-x^2})
\]
\[
= -2 \left( n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} + x \frac{d^n}{dx^n} e^{-x^2} \right)
\]
Rearranging this, we get:
\[
2x \frac{d^n}{dx^n} e^{-x^2/2} + \frac{d^{n+1}}{dx^{n+1}} e^{-x^2/2} = -2n \frac{d^{n-1}}{dx^{n-1}} e^{-x^2/2}.
\]
Break up the first term into two pieces and multiply both sides by \((-1)^n e^{x^2/2}\) and get
\[
(-1)^n xe^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} + \left( (-1)^n xe^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} + (-1)^n e^{x^2/2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2/2} \right)
\]
\[
= (-1)^{n+1} 2n e^{x^2/2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2/2}.
\]
Last, note that \((-1)^{n+1} = (-1)^{n-1}\) and recognize \(h_n, h'_n\) and \(h_{n-1}\) to conclude
\[
xh_n(x) + h'_n(x) = 2nh_{n-1}(x)
\]
and we’re done. We’ve certainly learned much more than this last identity along the way!

(e) According to part (d),
\[
xh_n(x) + h'_n(x) = 2nh_{n-1}(x).
\]
Rewrite this, replacing \(n\) by \(n+1\), as
\[
xh_{n+1}(x) + h'_{n+1}(x) = 2(n+1)h_n(x).
\]
Now substitute equation (*)
\[
h_{n+1}(x) = xh_n(x) - h'_n(x)
\]
for both occurrences of \(h_{n+1}\) in the previous equation and get
\[
x(xh_n - h'_n) + (xh_n - h'_n)' = 2(n+1)h_n
\]
or
\[
(x^2h_n - xh'_n) + (h_n + xh'_n - h''_n) = 2(n+1)h_n
\]
which can be algebraically rearranged to
\[
h''_n - x^2h_n + (2n+1)h_n = 0
\]
which is what we are trying to show.

(f) We’ve done a few of these in the past, the new wrinkle is that the interval of integration is the whole real line from \(-\infty\) to \(\infty\). But the procedure is the same. For \(n \neq m\), we know that
\[
h''_n - x^2h_n = -(2n+1)h_n
\]
and

\[ h''_m - x^2 h_m = -(2m + 1) h_m. \]

We have

\[-(2n + 1) \langle h_n, h_m \rangle = \langle -(2n + 1)h_n, h_m \rangle\]
\[= \langle h''_n - x^2 h_n, h_m \rangle\]
\[= \int_{-\infty}^{\infty} (h''_n(x) - x^2 h_n(x)) h_m(x) \, dx\]
\[= \int_{-\infty}^{\infty} h''_n(x) h_m(x) \, dx - \int_{-\infty}^{\infty} x^2 h_n(x) h_m(x) \, dx\]
\[= h'_n(x) h_m(x) \bigg|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} h_n(x) h''_m(x) \, dx - \int_{-\infty}^{\infty} x^2 h_n(x) h_m(x) \, dx\]
\[= \int_{-\infty}^{\infty} h_n(x) (h''_m(x) - x^2 h_m(x)) \, dx\]
\[= \langle h_n, h''_m - x^2 h_m \rangle\]
\[= \langle h_n, -(2m + 1) h_m \rangle\]
\[= -(2m + 1) \langle h_n, h_m \rangle\]

where in the middle of this we integrated by parts twice (once with \( u = h_m \) and \( dv = h'_n \, dx \) and the second time with \( u = h_n \) and \( dv = h'_m \, dx \)) and used the fact that, because \( h_n \) and \( h_m \) are polynomials times \( e^{-x^2/2} \), they and their derivatives approach zero at both \( \infty \) and \( -\infty \), so the evaluation terms are zero.

And since we have shown that \( (2n + 1) \langle h_n, h_m \rangle = (2m + 1) \langle h_n, h_m \rangle \) with \( n \neq m \), it must be the case that \( \langle h_n, h_m \rangle = 0 \).